ZASSENHAUS AND LOWER CENTRAL FILTRATIONS OF PRO-p GROUPS CONSIDERED AS MODULES

by

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Abstract. — The goal of this paper is to study the action of groups on Zassenhaus and lower central filtrations of finitely generated pro-p groups. We shall focus on the semisimple case. Particular attention is given to finitely presented groups of cohomological dimension less than or equal to two.

Introduction

Context. — Let G be a finitely generated pro-p group, and denote by $\mathbb{A}$ the ring $\mathbb{Z}_p$ or $\mathbb{F}_p$. From $\mathbb{A}$, we recover some filtrations on G. Introduce $Al(\mathbb{A}, G) := \varprojlim \mathbb{A}[G/U]$, where U is an open normal subgroup of G, the completed group algebra of G over $\mathbb{A}$. Since $\mathbb{A}[G/U]$ is an augmented algebra over $\mathbb{A}$, then $Al(\mathbb{A}, G)$ is also. Consequently, we denote by $Al_n(\mathbb{A}, G)$ the n-th power of the augmentation ideal of $Al(\mathbb{A}, G)$. Define:

$$G_n(\mathbb{A}) := \{g \in G; g - 1 \in Al_n(\mathbb{A}, G)\},$$

this is a filtration of G.

Observe that $\{G_n(\mathbb{F}_p)\}_{n\in\mathbb{N}}$ denotes the Zassenhaus filtration of G (see for instance [**21**]), and is an open characteristic basis of G. Similarly, under certain conditions (see [**16**]), the filtration $\{G_n(\mathbb{Z}_p)\}_{n\in\mathbb{N}}$ is equal to the lower central series of G, i.e. $G_1(\mathbb{Z}_p) = G$ and $G_{n+1}(\mathbb{Z}_p) = [G_n(\mathbb{Z}_p); G]$. When the context is clear, we omit to write $\mathbb{A}$ for filtrations (and future associated invariants). Our goal is to study the following Lie algebras:

$$\mathscr{L}(\mathbb{A}, G) := \bigoplus_{n\in\mathbb{N}} \mathscr{L}_n(\mathbb{A}, G), \quad \text{where} \quad \mathscr{L}_n(\mathbb{A}, G) := G_n(\mathbb{A})/G_{n+1}(\mathbb{A}), \quad \text{and}$$

$$\mathscr{E}(\mathbb{A}, G) := \bigoplus_{n\in\mathbb{N}} \mathscr{E}_n(\mathbb{A}, G), \quad \text{where} \quad \mathscr{E}_n(\mathbb{A}, G) := Al_n(\mathbb{A}, G)/Al_{n+1}(\mathbb{A}, G).$$

We always assume that $\mathscr{L}(\mathbb{A}, G)$ is **torsion-free** over $\mathbb{A}$. Notice that this condition is automatically satisfied when $\mathbb{A} := \mathbb{F}_p$, contrary to the case $\mathbb{A} := \mathbb{Z}_p$ (see for instance

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[**15**, Theorem] and [**14**, Théorème 2]). Since G is finitely generated, one defines for every integer n:

$$a_n(\mathbb{A}) := \operatorname{rank}_{\mathbb{A}} \mathscr{L}_n(\mathbb{A}, G), \quad \text{and} \quad c_n(\mathbb{A}) := \operatorname{rank}_{\mathbb{A}} \mathscr{E}_n(\mathbb{A}, G),$$
$$gocha(\mathbb{A}, t) := \sum_{n \in \mathbb{N}} c_n t^n.$$

The series $gocha(\mathbb{F}_p, t)$ was first introduced by Golod and Shafarevich in [**8**]. It allowed them to obtain information on class field towers over some fields (see for instance [**4**, Chapter IX]). Later in 1965, Lazard studied analytic pro-p groups in [**19**], i.e. Lie groups over $\mathbb{Q}_p$ (see [**19**, Définition 3.1.2]). Labute [**17**], also used the series $gocha(\mathbb{Z}_p, t)$ in order to study mild groups and their related properties.

Jennings, Lazard and Labute gave an explicit relation between $gocha$ and $(a_n)_{n \in \mathbb{N}}$ ([**19**, Proposition 3.10, Appendice A], and [**21**, Lemma 2.10]):

$$gocha(\mathbb{A}, t) = \prod_{n \in \mathbb{N}} P(\mathbb{A}, t^n)^{a_n(\mathbb{A})},$$
(1)
$$\text{where} \quad P(\mathbb{F}_p, t) := \left(\frac{1 - t^p}{1 - t}\right), \quad \text{and} \quad P(\mathbb{Z}_p, t) := \left(\frac{1}{1 - t}\right).$$

From Formula (1), Lazard deduced an alternative for the growth of $(c_n(\mathbb{F}_p))_{n \in \mathbb{N}}$ (for general references, see [**5**, Part 12.3]), this is [**19**, Théorème 3.11, Appendice A.3]:

***Theorem* (Alternative des Gocha).** — *We have the following alternative:*

- *Either G is an analytic pro-p group, so there exists an integer n such that $a_n(\mathbb{F}_p) = 0$ and the sequence $(c_n(\mathbb{F}_p))_{n \in \mathbb{N}}$ has polynomial growth with n.*
- *Or G is not an analytic pro-p group, then for every $n \in \mathbb{N}$, $a_n(\mathbb{F}_p) \neq 0$, and the sequence $(c_n(\mathbb{F}_p))_{n \in \mathbb{N}}$ does admit an exponential growth with n (i.e. grows faster than any polynomial in n).*

In 2016, Mináč, Rogelstad and Tân [**21**] improved Formula (1): they gave an explicit relation between the sequences $(a_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$. The main idea was to introduce the coefficients $b_n \in \mathbb{Q}$, namely defined by:

$$\log(gocha(\mathbb{A}, t)) := \sum_{n \in \mathbb{N}} b_n(\mathbb{A}) t^n.$$

They obtained the following formula ([**21**, Theorem 2.9 and Lemma 2.10]): if we write $n = mp^k$, with m coprime to p, then

$$a_n(\mathbb{F}_p) = w_m(\mathbb{F}_p) + w_{mp}(\mathbb{F}_p) + \cdots + w_{mp^k}(\mathbb{F}_p), \quad a_n(\mathbb{Z}_p) = w_n(\mathbb{Z}_p);$$
(2)
$$\text{where } w_n(\mathbb{A}) := \frac{1}{n} \sum_{m | n} \mu(n/m) m b_m(\mathbb{A}) \quad \text{and} \quad \mu \text{ is the Möbius function.}$$

Notice that the number $w_n(\mathbb{F}_p)$ (resp. $c_n(\mathbb{Z}_p), a_n(\mathbb{Z}_p)$) is denoted by $w_n(G)$ (resp. $d_n(G)$, $e_n(G)$) in [**21**, Part 2]. Furthermore, Mináč, Rogelstad and Tân asked the following question, [**21**, Question 2.13]:

Do we have $c_n(\mathbb{F}_p) := c_n(\mathbb{Z}_p)$?

Theorem 3.5 answers this question positively when G is finitely presented and $\operatorname{cd}(G) \leqslant 2$. To proceed, we compute $(c_n(\mathbb{A}))_{n \in \mathbb{N}}$ by the Lyndon resolution (see

[**3**, Corollay 5.3]), and as a consequence, we infer an explicit formula for $a_n(\mathbb{A})$ using Formula (2). Weigel ([**27**, Theorem D]) also gave a different formula from (2), involving $a_n(\mathbb{Z}_p)$ and roots of $1/gocha(\mathbb{Z}_p, t)$.

Statement of main results. — The goal of this paper is to extend equations (1), (2) and Gocha's alternative in an equivariant context. We use here the terminology equivariant to stress the action of groups.

Let q be a prime dividing $p-1$, and assume that $Aut(G)$ contains a cyclic subgroup Δ of order q. We denote by $\mathrm{Irr}(\Delta)$ the group of $\mathbb{A}$-irreducible characters χ of Δ, with trivial character $\mathbb{1}$: this is a group of order q which does not depend on the choice of $\mathbb{A}$ (for general references on $\mathbb{A}$-characters, see [**24**, Chapitre 14]). If M is a $\mathbb{A}[\Delta]$-module, one defines the eigenspaces of M by:

$$M[\chi] := \{x \in M; \quad \forall \delta \in \Delta, \quad \delta(x) = \chi(\delta)x\}.$$

Notice that $\mathscr{L}_n(\mathbb{A}, G)$ and $\mathscr{E}_n(\mathbb{A}, G)$ are free, finitely generated over $\mathbb{A}$ and are $\mathbb{A}[\Delta]$-modules. We study the following quantities:

$$a_n^\chi(\mathbb{A}) := \mathrm{rank}_{\mathbb{A}} \mathscr{L}_n(\mathbb{A}, G)[\chi], \quad \text{and} \quad c_n^\chi(\mathbb{A}) := \mathrm{rank}_{\mathbb{A}} \mathscr{E}_n(\mathbb{A}, G)[\chi].$$

From Maschke's Theorem and [**24**, Partie 14.4], we obtain the following equality:

$$a_n(\mathbb{A}) = \sum_{\chi \in \mathrm{Irr}(\Delta)} a_n^\chi(\mathbb{A}), \quad \text{and} \quad c_n(\mathbb{A}) = \sum_{\chi \in \mathrm{Irr}(\Delta)} c_n^\chi(\mathbb{A}).$$

This article has three parts.

Part 1 is mostly inspired by arguments given in [**21**]. Denote by $R[\Delta]$ the finite representation ring of Δ over $\mathbb{A}$. Observe that $R[\Delta]$ is a ring isomorphic to $\mathbb{Z}[\mathrm{Irr}(\Delta)]$, consequently $R[\Delta] \bigotimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$-algebra isomorphic to $\mathbb{Q}[\mathrm{Irr}(\Delta)]$. Instead of considering series with coefficients in $\mathbb{Q}$, as Filip [**6**] and Stix [**25**] did, we study series with coefficients in $R[\Delta] \bigotimes_{\mathbb{Z}} \mathbb{Q}$. Let us introduce:

$$gocha^*(\mathbb{A}, t) := \sum_{n \in \mathbb{N}} \left(\sum_{\chi \in \mathrm{Irr}(\Delta)} c_n^\chi(\mathbb{A}) \chi \right) t^n.$$

We infer an equivariant version of the equality (1):

Theorem A. — *The following equality holds for series with coefficients in $R[\Delta]$:*

$$gocha^*(\mathbb{A}, t) = \prod_{n \in \mathbb{N}} \prod_{\chi \in \mathrm{Irr}(\Delta)} P_\chi(\mathbb{A}, t^n)^{a_n^\chi(\mathbb{A})},$$

$$\textit{where} \quad P_\chi(\mathbb{F}_p, t) := \frac{1 - \chi.t^p}{1 - \chi.t}, \quad \textit{and} \quad P_\chi(\mathbb{Z}_p, t) := \frac{1}{1 - \chi.t}$$

As done in Part 2 [**21**], one introduces the logarithm of series with coefficients in $R[\Delta]$, defined by rationals $b_n^\chi(\mathbb{A}) \in \mathbb{Q}$:

$$log(gocha^*(\mathbb{A}, t)) := \sum_{n \in \mathbb{N}} \left(\sum_{\chi \in \mathrm{Irr}(\Delta)} b_n^\chi(\mathbb{A}) \chi \right) t^n.$$

Then, we obtain an equivariant version of Formula (2).

Write $n := mp^k$, with $(m,p)=1$, and assume $(n,q)=1$. Then:

$$a_n^{\chi}(\mathbb{F}_p) = w_m^{\chi}(\mathbb{F}_p) + w_{mp}^{\chi}(\mathbb{F}_p) + \cdots + w_{mp^k}^{\chi}(\mathbb{F}_p), \quad \text{and} \quad a_n^{\chi}(\mathbb{Z}_p) = w_n^{\chi}(\mathbb{Z}_p),$$
$$\text{where} \quad w_n^{\chi}(\mathbb{A}) := \frac{1}{n}\sum_{m|n} \mu(n/m) m b_m^{\chi^{m/n}}(\mathbb{A}) \in \mathbb{Q}. \tag{3}$$

Some results on the coefficients $a_n^{\chi}(\mathbb{Z}_p)$ were given by Filip [**6**] and Stix [**25**] for groups defined by one quadratic relation.

In Part 2, we study cardinalities of eigenspaces of $\mathscr{L}(\mathbb{A},G)$. When $\mathscr{L}(\mathbb{A},G)$ is infinite dimensional (as a free module over $\mathbb{A}$), we observe using the pigeonhole principle that $\mathscr{L}(\mathbb{A},G)$ admits at least one infinite dimensional eigenspace.

Main Question: Which eigenspaces of $\mathscr{L}(\mathbb{A},G)$ are infinite dimensional?

For this purpose, we introduce χ_0-filtration on $Al(\mathbb{A},G)$, which depends on a fixed non-trivial irreducible character χ_0. It is denoted by $(E_{\chi_0,n}(\mathbb{A},G))_n$, and described in Subpart 2.1. Furthermore, we assume that $E_{\chi_0,n}(\mathbb{A},G)/E_{\chi_0,n+1}(\mathbb{A},G)$ is **torsion-free** over $\mathbb{A}$. This condition is automatically satisfied when $\mathbb{A}=\mathbb{F}_p$; and for $\mathbb{A}=\mathbb{Z}_p$, it is true whenever G is free or in the situation of [**14**, Théorème 2]. This allows us to define $gocha_{\chi_0}(\mathbb{A},t)$ by:

$$gocha_{\chi_0}(\mathbb{A},t) := \sum_{n\in\mathbb{N}} c_{\chi_0,n}(\mathbb{A})t^n,$$
$$\text{where} \quad c_{\chi_0,n}(\mathbb{A}) := \operatorname{rank}_{\mathbb{A}}(E_{\chi_0,n}(\mathbb{A},G)/E_{\chi_0,n+1}(\mathbb{A},G)).$$

Part 3 illustrates our theoretical results for finitely presented pro-p groups G, with cohomological dimension $\operatorname{cd}(G)$ less than or equal to 2.

Proposition 3.3 allows us to compute the *gocha* series of G, and shows that the inverse of the gocha series is a polynomial. Then Theorem 3.5 answers (and generalizes) [**21**, Question 2.13], showing that $gocha(\mathbb{A},t)$, $gocha^*(\mathbb{A},t)$ and $gocha_{\chi_0}(\mathbb{A},t)$ do not depend on the choice of the ring $\mathbb{A}$. Finally, considering [**27**, Theorem D] in our context, one recovers a_n^{χ} from roots of the polynomial $1/gocha^*$ (see Proposition 3.8).

Let us now introduce our last result. Since (Proposition 3.3) $\chi_{eul,\chi_0}(t) := 1/gocha_{\chi_0}(t)$ is a polynomial, we write the degree of χ_{eul,χ_0} as $\deg_{\chi_0}(G)$. Denote the χ_0-eigenvalues of G by $\lambda_{\chi_0,i}$, and let $L_{\chi_0}(G)$ be the χ_0-entropy of G defined by:

$$\chi_{eul,\chi_0}(t) := \prod_{i=1}^{\deg_{\chi_0}(G)} (1-\lambda_{\chi_0,i}t), \quad L_{\chi_0}(G) := \max_{1\leqslant i\leqslant \deg_{\chi_0}(G)} |\lambda_{\chi_0,i}|.$$

Theorem B. — *Assume for some χ_0 that $L_{\chi_0}(G)$ is reached for a unique eigenvalue $\lambda_{\chi_0,i}$ such that:*

(i) $\lambda_{\chi_0,i}$ *is real,*
(ii) $L_{\chi_0}(G) = \lambda_{\chi_0,i} > 1$.

Then every eigenspace of $\mathscr{L}(\mathbb{A},G)$ is infinite dimensional.

We also prove in Theorem 3.12, that every finitely generated noncommutative free pro-p group G satisfies the hypotheses of Theorem B. Let us finish this introduction with explicit examples:

Example 1 **(Cohomological dimension** 2**).** — *Let us take* $p = 103$*,* $q = 17$*. Observe that* $\overline{8} \in \mathbb{F}_{103}$ *is a primitive* 17*-th root of unity.*
*Consider the pro-*103 *group* G*, generated by three generators* x, y, z *and one relation* $u = [x; y]$*. By* [**15**, Theorem]*, the* $\mathbb{Z}_p$*-module* $\mathscr{L}(\mathbb{Z}_p, G)$ *is torsion-free. If we apply* [**7**, Corollary 5.3] *and Proposition 3.3, we remark that* $\mathrm{cd}(G) = 2$ *and:*

$$gocha(\mathbb{A}, t) := 1/(1 - 3t + t^2).$$

Introduce an automorphism δ *on* G*, by* $\delta(x) := x^8$*,* $\delta(y) := y^{8^2}$ *and* $\delta(z) := z^{8^3}$*; Proposition 3.16 justifies that this action is well defined. Consequently* $\Delta := \langle \delta \rangle$ *is a subgroup of order* 17 *of* $Aut(G)$*. Fix the character* $\chi_0 \colon \Delta \to \mathbb{F}_{103}^{\times}; \delta \mapsto \overline{8}$*.*

Applying Formula (3)*, let us compute some coefficients* a_n^{χ} *and* c_n^{χ}*.*
Observe first that:

$$gocha^*(\mathbb{A}, t) := \frac{1}{1 - ({\chi_0} + {\chi_0}^2 + {\chi_0}^3).t + {\chi_0}^3.t^2}, \quad and$$

$$\log(gocha^*(\mathbb{A}, t)) = ({\chi_0} + {\chi_0}^2 + {\chi_0}^3).t + ({\chi_0}^6/2 + {\chi_0}^5 + \frac{3\chi_0^4}{2} + \frac{\chi_0^2}{2}).t^2 + (\frac{\chi_0^9}{3} + {\chi_0}^8 + 2{\chi_0}^7 + \frac{4\chi_0^6}{3} + \chi_0^5 + \frac{\chi_0^3}{3}).t^3 + \dots.$$

From Formula (2)*, we infer:* $a_2 = 2$ *and* $a_3 = 5$*. Furthermore Formula* (3) *gives us:*

$$a_2^{\chi_0^i} = \frac{2b_2^{\chi_0^i} - b_1^{\chi_0^{9i}}}{2}, \quad and \quad a_3^{\chi_0^i} = \frac{3b_3^{\chi_0^i} - b_1^{\chi_0^{6i}}}{3}.$$

Consequently, we obtain:

- $a_2^{\chi_0^4} = a_2^{\chi_0^5} = 1$*, else* $a_2^{\chi_0^i} = 0$ *when* $i \neq 5$*.*
- $a_3^{\chi_0^8} = a_3^{\chi_0^6} = a_3^{\chi_0^5} = 1$*, and* $a_3^{\chi_0^7} = 2$*. Else if* $i \notin \{5; 6; 7; 8\}$*,* $a_3^{\chi_0^i} = 0$*.*

Here, by [**16**, Theorem 1 and Part 3]*, the algebra* $\mathscr{L}_{\chi_0}(G, \mathbb{Z}_p)$ *is torsion-free over* $\mathbb{Z}_p$*.*
We have:

$$gocha_{\chi_0}(\mathbb{A}, t) := \frac{1}{1 - t - t^2},$$

and the maximal χ_0*-eigenvalue of* G *is real and strictly greater than* 1*.*
Therefore by Theorem B, all eigenspaces of $\mathscr{L}(\mathbb{A}, G)$ *are infinite dimensional.*

Example 2 **(FAB example).** — *Following arguments given by* [**9**]*, we enrich the example given in* [**11**, Part 2.1]*, and obtain an example where* G *is FAB, i.e. every open subgroup has finite abelianization (for more details, see Example 3.20, and for references on FAB groups, see* [**17**] *and* [**20**]*).*

Take $p = 3$*, and consider* $\mathrm{K} := \mathbb{Q}(\sqrt{-163})$*. Then we define* $\Delta := \mathrm{Gal}(\mathrm{K}/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$*, and fix* χ_0 *the non-trivial irreducible character of* Δ *over* $\mathbb{F}_p$*. Consider the following set of places in* $\mathbb{Q}$*:* $\{7, 19, 13, 31, 337, 43\}$*. The class group of* K *is trivial, the primes* $7, 19, 13, 31, 337$ *are inert in* K*, and the prime* 43 *totally splits in* K*.*

Define S *the primes above the previous set in* K*, and* K_S *the maximal* p*-extension unramified outside* S*. Then* Δ *acts on* $G := \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$*, which is FAB by Class Field Theory.*

We can show that the pro-p group G *is mild, and Proposition 3.18 gives:*

$$gocha^*(\mathbb{F}_p, t) := \frac{1}{1 - (6 + \chi_0)t + (6 + \chi_0)t^2}, \quad and \quad gocha_{\chi_0}(\mathbb{F}_p, t) := \frac{1}{1 - t - 5t^2 + 6t^4}.$$

Therefore by Theorem B, all eigenspaces of $\mathscr{L}(\mathbb{F}_p, G)$ are infinite dimensional.

Notations. — We follow the notations and definitions of [**1**] and [**19**, Appendice A].

Let p be an odd prime, and G a finitely generated pro-p group with minimal presentation $1 \to R \to F \to G \to 1$, and denote by $\mathbb{A}$ one of the rings $\mathbb{F}_p$ or $\mathbb{Z}_p$. Assume that $Aut(G)$ contains a cyclic subgroup Δ of order q, a prime factor of $p-1$. By [**10**, Lemma 2.15], we observe that Δ lifts to a subgroup of $Aut(F)$.

When the context is clear, we omit the $\mathbb{A}$ when denoting filtrations (and associated invariants). Additionally, we always suppose that $\mathscr{L}(\mathbb{A}, G)$ is **torsion-free over** $\mathbb{A}$.

Denote by $Al(\mathbb{A}, G)$ the completed group algebra of G over $\mathbb{A}$ and observe that G embeds naturally into $Al(\mathbb{A}, G)$.

For $\chi \in \mathrm{Irr}(\Delta)$, we fix $\{x_j^\chi\}_{1 \leqslant j \leqslant d^\chi}$ a lift in F of a basis of $\mathscr{L}_1(\mathbb{A}, G)[\chi]$, where $d^\chi := \mathrm{rank}_{\mathbb{A}} \mathscr{L}_1(\mathbb{A}, G)[\chi]$; by [**24**, Corollaire 3, Proposition 42, Chapitre 14], this basis does not depend on the choice of $\mathbb{A}$. The Magnus isomorphism, from [**19**, Chapitre II, Partie 3], gives us the following identification of $\mathbb{A}$-algebras between $Al(\mathbb{A}, F)$ and the noncommuative series over X_j^χ's with coefficients in $\mathbb{A}$:

$$\phi_{\mathbb{A}} \colon Al(\mathbb{A}, F) \simeq \mathbb{A}\langle\langle X_j^\chi; \chi \in \mathrm{Irr}(\Delta), 1 \leqslant j \leqslant d^\chi \rangle\rangle; \quad x_j^\chi \mapsto X_j^\chi + 1 \tag{4}$$

Define $E(\mathbb{A})$ as the algebra $\mathbb{A}\langle\langle X_j^\chi; \chi \in \mathrm{Irr}(\Delta), 1 \leqslant j \leqslant d^\chi \rangle\rangle$ filtered by $\deg(X_j^\chi) = 1$ and write $\{E_n(\mathbb{A})\}_{n \in \mathbb{N}}$ for its filtration. One introduces $I(\mathbb{A}, R)$ the ideal of $E(\mathbb{A})$ generated by $\{\phi_{\mathbb{A}}(r-1); r \in R\}$ endowed with the induced filtration $\{I_n(\mathbb{A}, R) := I(\mathbb{A}, R) \cap E_n(\mathbb{A})\}_{n \in \mathbb{N}}$, and $E(\mathbb{A}, G)$ the quotient filtered algebra $E(\mathbb{A})/I(\mathbb{A}, R)$, with induced filtration $\{E_n(\mathbb{A}, G)\}_{n \in \mathbb{N}}$.

We call $M := \bigoplus_{n \in \mathbb{N}} M_n$ a graded locally finite ($\mathbb{A}[\Delta]$-)module, if M_n is a finite dimensional ($\mathbb{A}[\Delta]$)-module for every integer n; and denote its Hilbert series by:

$$M(t) := \sum_{n \in \mathbb{N}} (\mathrm{rank}_{\mathbb{A}} M_n) t^n.$$

We make the following convention; we say that M is an $\mathbb{A}$-Lie algebra if M is a graded locally finite Lie algebra over $\mathbb{A}$, and when $\mathbb{A} := \mathbb{F}_p$ we assume in addition that M is a restricted p-Lie algebra. Recall the following graded locally finite $\mathbb{A}[\Delta]$-module and $\mathbb{A}$-Lie algebra, defined at the beginning:

$$\mathscr{E}(\mathbb{A}) := \bigoplus_{n \in \mathbb{N}} \mathscr{E}_n(\mathbb{A}), \quad \text{where} \quad \mathscr{E}_n(\mathbb{A}) := E_n(\mathbb{A})/E_{n+1}(\mathbb{A}),$$
$$\mathscr{L}(\mathbb{A}, G) := \bigoplus_{n \in \mathbb{N}} \mathscr{L}_n(\mathbb{A}, G), \quad \text{and} \quad \mathscr{E}(\mathbb{A}, G) := \bigoplus_{n \in \mathbb{N}} \mathscr{E}_n(\mathbb{A}, G).$$

If $P := \sum_{n \in \mathbb{N}} p_n t^n$ and $Q := \sum_{n \in \mathbb{N}} q_n t^n$ are two series with real coefficients, we say that : $P \leqslant Q \iff \forall n \in \mathbb{N}, \quad p_n \leqslant q_n$. We denote by μ the Möbius function.

1. An equivariant version of Mináč-Rogelstad-Tân's results

Recall:

$$gocha^*(\mathbb{A}, t) := \sum_{n \in \mathbb{N}} \left(\sum_{\chi \in \mathrm{Irr}(\Delta)} c_n^\chi \chi \right) t^n \in R[\Delta][[t]],$$

where $R[\Delta]$ denotes the finite representation ring of Δ (over $\mathbb{A}$).

1.1. Equivariant Hilbert series. — The aim of this subpart is to prove the following formula:

$$\text{(5)} \quad gocha^*(\mathbb{A},t) = \prod_{n\in\mathbb{N}} \prod_{\chi\in\mathrm{Irr}(\Delta)} P_\chi(\mathbb{A},t^n)^{a_n^\chi},$$
$$\text{where} \quad P_\chi(\mathbb{F}_p,t) := \frac{1-\chi.t^p}{1-\chi.t}, \quad \text{and} \quad P_\chi(\mathbb{Z}_p,t) := \frac{1}{1-\chi.t}.$$

This is Theorem A defined in our introduction.

Definition 1.1. — Let $M := \bigoplus_{n\in\mathbb{N}} M_n$ be an $\mathbb{A}$-Lie algebra, graded locally finite $\mathbb{A}[\Delta]$-module, with basis $\{x_{n,1};\dots;x_{n,m_n}\}_{n\in\mathbb{N}}$, where $m_n := \mathrm{rank}_{\mathbb{A}} M_n$. We define:

- the graded locally finite module with basis given by words on $\{x_{n,j}\}_{n\in\mathbb{N};j\in[\![1;m_n]\!]}$ by:
$$\tilde{U}(M) := \bigoplus_{n\in\mathbb{N}} \tilde{U}(M)_n,$$
moreover, when $\mathbb{A} := \mathbb{F}_p$, we also assume that the p-restricted operation is compatible with the multiplicative structure of $\tilde{U}(M)$;
- the equivariant Hilbert series of M with coefficient in $R[\Delta]$ by:
$$M^*(t) := \sum_{n\in\mathbb{N}} \left(\sum_{\chi\in\mathrm{Irr}(\Delta)} m_n^\chi \chi \right) t^n$$
$$\text{where} \quad m_n^\chi := \mathrm{rank}_{\mathbb{A}} M_n[\chi] \quad \text{for every integer } n.$$

Remark 1.2. — Since the action of Δ over a graded locally finite module is semi-simple, it always preserves the grading. Consequently, if M is a graded locally finite $\mathbb{A}[\Delta]$-module, then the graded locally finite module $\tilde{U}(M)$ is also endowed with a natural structure of graded locally finite $\mathbb{A}[\Delta]$-module.

We give a well-known result on Lie algebras, telling us that $\tilde{U}$ is a universal enveloping algebra of M.

Theorem 1.3 **(Poincaré-Birkhoff-Witt)**. — *Let M be a graded locally finite $\mathbb{A}[\Delta]$-module and $\mathbb{A}$-Lie algebra. Then $\tilde{U}(M)$ is a graded locally finite $\mathbb{A}[\Delta]$-module, universal $\mathbb{A}$-Lie algebra of M.*

Proof. — When $\mathbb{A} := \mathbb{Z}_p$, see for instance [**17**, Theorem 2.1].
When $\mathbb{A} := \mathbb{F}_p$, see for instance [**5**, Proposition 12.4]. □

Corollary 1.4. — *The set $\mathscr{E}(\mathbb{A},G)$ is a graded locally finite, $\mathbb{A}$-universal Lie algebra of $\mathscr{L}(\mathbb{A},G)$. Consequently $\mathscr{E}(\mathbb{A},G)$ is torsion-free.*

Proof. — Let us first prove that $\mathscr{E}(\mathbb{A},G)$ is a graded locally finite, $\mathbb{A}$-universal Lie algebra of $\mathscr{L}(\mathbb{A},G)$. By Theorem 1.3, we only need to show that $\tilde{U}(\mathscr{L}(\mathbb{A},G)) \simeq \mathscr{E}(\mathbb{A},G)$.

For $\mathbb{A} := \mathbb{F}_p$, see [**19**, Appendice A, Théorème 2.6].

For $\mathbb{A} := \mathbb{Z}_p$, the proof of [**12**, Theorem 1.3] carries to the case $E(\mathbb{Z}_p,G)$ with minor alterations. We consider $\mathbb{Z}_p$ and $\mathbb{Q}_p$ rather than $\mathbb{Z}$ and $\mathbb{Q}$. Furthermore, we conclude using the fact that G is finitely generated, so $\mathrm{Grad}(E(\mathbb{Z}_p,G)) = \mathscr{E}(\mathbb{Z}_p,G)$ is isomorphic to $\mathrm{Grad}(\mathbb{Z}_p[G])$, where $\mathbb{Z}_p[G]$ is filtered by power of the augmentation ideal over $\mathbb{Z}_p$. □

Remark 1.5. — Notice that $\mathscr{E}(\mathbb{A},G)$ is also isomorphic to $\tilde{U}(\mathscr{L}(\mathbb{A},G))$ as an $\mathbb{A}[\Delta]$-module. Therefore, we have:

$$\tilde{U}(\mathscr{L}(\mathbb{A},G))^*(t) := gocha^*(\mathbb{A},t).$$

Before proving Formula 5, we need the following result:

Lemma 1.6. — *Let M be a graded locally finite $\mathbb{A}[\Delta]$-module and $\mathbb{A}$-Lie algebra, then:*

$$\tilde{U}(M)^*(t) = \prod_{n\in\mathbb{N}} \prod_{\chi\in\mathrm{Irr}(\Delta)} P_\chi(\mathbb{A},t^n)^{m_n^\chi},$$

$$\textit{where} \quad P_\chi(\mathbb{F}_p,t) := \frac{1-\chi.t^p}{1-\chi.t}, \quad \textit{and} \quad P_\chi(\mathbb{Z}_p,t) := \frac{1}{1-\chi.t}.$$

Proof. — Let us first prove the case $\mathbb{A} := \mathbb{F}_p$.

We are inspired by the proof of [**5**, Corollary 12.13]. Observe that if M and N are graded locally finite $\mathbb{F}_p[\Delta]$-modules, then $M \bigotimes_{\mathbb{F}_p} N$ is also a graded locally finite $\mathbb{F}_p[\Delta]$-module; moreover $(M \bigotimes_{\mathbb{F}_p} N)^*(t) := M^*(t)N^*(t)$, and $\tilde{U}(M \bigoplus N) = \tilde{U}(M) \bigotimes_{\mathbb{F}_p} \tilde{U}(N)$. So assume that :

$$M^*(t) := \sum_n m_n \chi_0.t^n, \qquad \text{for some fixed and non-trivial } \chi_0 \in \mathrm{Irr}(\Delta).$$

Consider $X_n := \{x_{n,1},\dots,x_{n,m_n}\}$, an $\mathbb{F}_p[\Delta]$-basis of M_n, where each $x_{n,j}$ is of degree n. Then a graded locally finite $\mathbb{F}_p[\Delta]$-basis of M is given by the (disjoint) union of all X_n's. Denote by

$$\tilde{U}(M)^*(t) := \sum_{r\in\mathbb{N}} \left(\sum_{\chi\in\mathrm{Irr}(\Delta)} u_r^\chi \chi \right) t^r, \text{ where } u_r^\chi := \dim_{\mathbb{F}_p} \tilde{U}(M)_r[\chi].$$

We need to compute $u_r^{\chi_0^i}$, where $i \in \mathbb{Z}/q\mathbb{Z}$: this is the number of products of the form

$$\prod_{n=1}^{r} \prod_{j=1}^{m_n} (x_{n,j}\chi_0)^{m_{n,j}}, \quad \text{where} \quad 0 \leqslant m_{n,j} \leqslant p-1,$$

such that

$$\sum_{n=1}^{r} \sum_{j=1}^{m_n} n m_{n,j} = r \quad \text{and} \quad \sum_{n=1}^{r} \sum_{j=1}^{m_n} m_{n,j} \equiv i \pmod q.$$

Notice that the coefficient before t^r of the polynomial

$$\prod_{n=1}^{r} [1 + \chi_0 t^n + \cdots + \chi_0^{p-1} t^{(p-1)n}]^{m_n}$$

is

$$\sum_{n=1}^{r} \left(\sum_{j=1}^{m_n} {\chi_0}^{m_{n,j}} \right) t^r, \quad \text{where} \quad 0 \leqslant m_{n,j} \leqslant p-1, \quad \text{and} \quad \sum_{n=1}^{r} \sum_{j=1}^{m_n} n m_{n,j} = r.$$

Consequently the coefficient before $\chi_0^i t^r$ is exactly $u_r^{\chi_0^i}$.

Let us now prove the case $\mathbb{A} := \mathbb{Z}_p$.

By the Poincaré-Birkhoff-Witt Theorem, the set $\tilde{U}(M)$ is the symmetric Lie algebra over M. Similarly to the previous case, we just need to study the case where there exists a unique χ_0 such that $M^*(t) := \sum_n m_n \chi_0.t^n$. We get:

$$\tilde{U}(M)^*(t) = \prod_n \left(\frac{1}{1-\chi_0.t}\right)^{m_n^\chi}.$$

One deduces the general case. □

Proof of Formula (5). — We apply Lemma 1.6 and Corollary 1.4 to obtain:

$$gocha^*(\mathbb{A}, t) = \prod_{n\in\mathbb{N}} \prod_{\chi\in\mathrm{Irr}(\Delta)} P_\chi(\mathbb{A}, t^n)^{a_n^\chi},$$

$$\text{where} \quad P_\chi(\mathbb{F}_p, t) := \frac{1-\chi.t^p}{1-\chi.t}, \quad \text{and} \quad P_\chi(\mathbb{Z}_p, t) := \frac{1}{1-\chi.t}$$

□

1.2. Proof of Formula (3). — The aim of this part is to prove the following Proposition:

Proposition 1.7. — *Write $n = mp^k$ with $(m,p) = 1$ and $(n,q) = 1$, then:*

$$a_n^\chi(\mathbb{F}_p) = w_m^\chi(\mathbb{F}_p) + w_{mp}^\chi(\mathbb{F}_p) + \cdots + w_{mp^k}^\chi(\mathbb{F}_p), \quad \textit{and} \quad a_n^\chi(\mathbb{Z}_p) = w_n^\chi(\mathbb{Z}_p);$$

$$\textit{where} \quad w_n^\chi := \frac{1}{n}\sum_{m|n} \mu(n/m) m b_m^{\chi^{m/n}} \in \mathbb{Q}.$$

This is Formula (3) given in our introduction.

The strategy of the proof is to transform the product formula given by (5), into a sum in $(R[\Delta]\otimes_{\mathbb{Z}}\mathbb{Q})[[t]]$.

Definition 1.8 (log **function**). — If $P \in 1 + tR[\Delta][[t]]$, we define:

$$\log(P)(t) := -\sum_n \frac{(1-P(t))^n}{n} \in (R[\Delta]\otimes_{\mathbb{Z}}\mathbb{Q})[[t]].$$

Remark 1.9. — Note that the log function enjoys the following properties:

(i) If P and Q are in $1 + tR[\Delta][[t]]$, then:

$$\log(PQ) = \log(P) + \log(Q), \quad \text{and}$$

$$\log(1/P) = -\log(P).$$

(ii) If u is in $tR[\Delta][[t]]$, then

$$\log\left(\frac{1}{1-u}\right) = \sum_{\nu=1}^{\infty} \frac{u^\nu}{\nu}.$$

Define the sequence $(b_n^\chi(\mathbb{A}))_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ by:

$$\log(gocha^*(\mathbb{A}, t)) = \sum_{n\geqslant 1}\left(\sum_{\chi\in\mathrm{Irr}(\Delta)} b_n^\chi(\mathbb{A})\chi\right)t^n.$$

Proposition 1.10. — *If* $(n,q)=1$, *we infer:*

$$b_n^{\chi^n}(\mathbb{F}_p) := \frac{1}{n}\left(\sum_{m|n} m a_m^{\chi^m}(\mathbb{F}_p) - \sum_{rp|n} rp a_r^{\chi^r}(\mathbb{F}_p)\right), \quad \textit{and} \quad b_n^{\chi^n}(\mathbb{Z}_p) := \frac{1}{n}\sum_{m|n} m a_m^{\chi^m}(\mathbb{Z}_p).$$

Proof. — Let us just prove the case $\mathbb{A} := \mathbb{F}_p$ (the case $\mathbb{A} := \mathbb{Z}_p$ is similar).

First, Formula (5) gives us:

$$gocha^*(\mathbb{F}_p, t) = \prod_{n\in\mathbb{N}} \prod_{\chi\in\mathrm{Irr}(\Delta)} \left(\frac{1-\chi.t^{np}}{1-\chi.t^n}\right)^{a_n^\chi}.$$

Let us take the logarithm to obtain:

$$\log(gocha^*(\mathbb{F}_p, t)) = \sum_n \sum_{\chi\in\mathrm{Irr}(\Delta)} a_n^\chi \left[\log(1-(\chi.t^n)^p) - \log(1-\chi.t^n)\right],$$

so that

$$\sum_{n\in\mathbb{N}}\left(\sum_{\chi\in\mathrm{Irr}(\Delta)} b_n^\chi \chi\right) t^n = \sum_{w=1}^{\infty}\sum_{\chi\in\mathrm{Irr}(\Delta)} a_w^\chi \left(\sum_{\nu=1}^{\infty}\frac{(\chi.t^w)^\nu}{\nu} - \sum_{r=1}^{\infty}\frac{(\chi.t^w)^{rp}}{r}\right),$$

from which we conclude

$$\sum_{n=1}^{\infty} n\left(\sum_{\chi\in\mathrm{Irr}(\Delta)} b_n^\chi \chi\right) t^n = \sum_{n=1}^{\infty}\left(\sum_{\chi\in\mathrm{Irr}(\Delta)} (\sum_{m|n} m a_m^\chi \chi^{n/m} - \sum_{rp|n} rp a_r^\chi \chi^{n/r})\right) t^n.$$

Then we infer:

$$n b_n^{\chi^n} = \sum_{m|n} m a_m^{\chi^m} - \sum_{rp|n} rp a_r^{\chi^r}.$$

□

Proof of Proposition 1.7. — Again, we just prove the case $\mathbb{A} := \mathbb{F}_p$.
We are inspired by the proof of [**21**, Theorem 2.9].
First, we assume $(n,p)=1$, then by Proposition 1.10, we obtain:

$$n b_n^{\chi^n} = \sum_{m|n} m a_m^{\chi^m}.$$

So, using the Möbius inversion Formula, we obtain:

$$a_n^{\chi^n} = w_n^{\chi^n}, \quad \text{thus} \quad a_n^\chi = w_n^\chi.$$

Now, let us assume p divides n. We show by induction on n that:

(∗) $$a_n^{\chi^n} = a_{n/p}^{\chi^{n/p}} + w_n^{\chi^n}$$

- If $n=p$, then by Proposition 1.10, we have: $p b_p^{\chi^p} = p a_p^{\chi^p} + a_1^\chi - p a_1^\chi$. So,

  $$p w_p^{\chi^p} = p b_p^{\chi^p} - b_1^\chi = p a_p^{\chi^p} - p a_1^\chi.$$

  Therefore, $a_p^{\chi^p} = a_1^\chi + w_p^{\chi^p}$.

- Let us fix n, an integer such that $p|n$, and assume equation $(*)$ is true for all m such that $m \neq n$ and $p|m|n$. Then, following Proposition 1.10, we have:

$$\begin{aligned}
nb_n^{\chi^n} &= \sum_{m|n} ma_m^{\chi^m} - \sum_{rp|n} rpa_r^{\chi^r} \\
&= \sum_{m|n;(m,p)=1} ma_m^{\chi^m} + \sum_{p|m|n} m\left(a_m^{\chi^m} - a_{m/p}^{\chi^{m/p}}\right) \\
&= \sum_{m|n;(m,p)=1} mw_m^{\chi^m} + \sum_{p|m|n;m\neq n} mw_m^{\chi^m} + n\left(a_n^{\chi^n} - a_{n/p}^{\chi^{n/p}}\right) \\
&= \sum_{m|n;m\neq n} mw_m^{\chi^m} + n\left(a_n^{\chi^n} - a_{n/p}^{\chi^{n/p}}\right).
\end{aligned}$$

Moreover, by the Möbius inversion formula, we have:

$$nb_n^{\chi^n} = \sum_{m|n} mw_m^{\chi^m}.$$

Therefore, we obtain:

$$nw_n^{\chi^n} = n\left(a_n^{\chi^n} - a_{n/p}^{\chi^{n/p}}\right).$$

□

Remark 1.11. — Formula (3) was already given for groups defined by one quadratic relation by Filip [**6**, Formula (4.7)] (for $\mathbb{C}$-representations in a geometrical context) and by Stix [**25**, Formula (14.16)] (in a Galois-theoretical context). Additionally, they computed explicitely the coefficients $b_n^{\chi}(\mathbb{Z}_p)$. We discuss this analogy in Theorem 3.8.

Remark 1.12. — Let us reformulate [**21**, Question 2.13], asked by Mináč-Rogelstad-Tân, in our equivariant context:

Do we have for every integer n and every irreducible character χ, the equality $c_n^{\chi}(\mathbb{Z}_p) = c_n^{\chi}(\mathbb{F}_p)$?

Later in this paper, we give a positive answer to this question, when G is finitely presented and $\operatorname{cd}(G) \leqslant 2$ (see Theorem 3.5).

2. Infinite dimensional eigenspaces of $\mathscr{L}(\mathbb{A}, G)$

The goal of this part is to study infinite dimensional eigenspaces (as a free $\mathbb{A}$-module) of

$$\mathscr{L}(\mathbb{A}, G) := \bigoplus_{n\in\mathbb{N}} \mathscr{L}_n(\mathbb{A}, G), \quad \text{where} \quad \mathscr{L}_n(\mathbb{A}, G) := G_n(\mathbb{A})/G_{n+1}(\mathbb{A}).$$

For this purpose, we introduce χ_0-filtrations.

2.1. Definition of χ_0-filtrations. — From now on, we make no distinction between $\mathbb{Z}/q\mathbb{Z}$ and the set $[\![1; q]\!]$. Observe the following isomorphism of groups, which depends on the choice of a fixed non-trivial irreducible character χ_0:

$$\psi_{\chi_0} \colon (\operatorname{Irr}(\Delta); \otimes) \to (\mathbb{Z}/q\mathbb{Z}; +); \quad \chi_0^i \mapsto i.$$

Recall that $\phi_{\mathbb{A}}$ denotes the Magnus' isomorphism introduced in (4). We define $E_{\chi_0}(\mathbb{A})$ as the $\mathbb{A}$-algebra $\mathbb{A}\langle\langle X_j^{\chi}; \chi \in \mathrm{Irr}(\Delta), 1 \leqslant j \leqslant d^{\chi}\rangle\rangle$ filtered by $\deg(X_j^{\chi}) = \psi_{\chi_0}(\chi)$, and $\{E_{\chi_0,n}(\mathbb{A})\}_{n\in\mathbb{N}}$ as its filtration: called the χ_0-filtration of $Al(\mathbb{A}, F)$. We introduce

$$\mathscr{E}_{\chi_0}(\mathbb{A}) := \bigoplus_{n\in\mathbb{N}} \mathscr{E}_{\chi_0,n}(\mathbb{A}), \quad \text{where} \quad \mathscr{E}_{\chi_0,n}(\mathbb{A}) := E_{\chi_0,n}(\mathbb{A})/E_{\chi_0,n+1}(\mathbb{A}).$$

Write $I_{\chi_0}(\mathbb{A}, R)$ for the two-sided ideal generated by $\{\phi_{\mathbb{A}}(r-1); r \in R\} \subset E_{\chi_0}(\mathbb{A})$, endowed with filtration $\{I_{\chi_0,n}(\mathbb{A}, R) := I_{\chi_0}(\mathbb{A}, R) \cap E_{\chi_0,n}(\mathbb{A})\}_{n\in\mathbb{N}}$; and $E_{\chi_0}(\mathbb{A}, G)$ the quotient filtered algebra $E_{\chi_0}(\mathbb{A})/I_{\chi_0}(\mathbb{A}, R)$.
Define the following $\mathbb{A}$-module:

$$\mathscr{E}_{\chi_0}(\mathbb{A}, G) := \bigoplus_{n\in\mathbb{N}} \mathscr{E}_{\chi_0,n}(\mathbb{A}, G), \quad \text{where} \quad \mathscr{E}_{\chi_0,n}(\mathbb{A}, G) := E_{\chi_0,n}(\mathbb{A}, G)/E_{\chi_0,n+1}(\mathbb{A}, G).$$

Introduce:

$$G_{\chi_0,n}(\mathbb{A}) := \{g \in G; \phi_{\mathbb{A}}(g-1) \in E_{\chi_0,n}(\mathbb{A}, G)\}, \quad \text{and}$$
$$\mathscr{L}_{\chi_0}(\mathbb{A}, G) := \bigoplus_{n\in\mathbb{N}} \mathscr{L}_{\chi_0,n}(\mathbb{A}, G), \quad \text{where} \quad \mathscr{L}_{\chi_0,n}(\mathbb{A}, G) := G_{\chi_0,n}(\mathbb{A})/G_{\chi_0,n+1}(\mathbb{A}).$$

We always assume that the $\mathbb{A}$-Lie algebra $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$ is **torsion-free** over $\mathbb{A}$.

Lemma 2.1. — *The set $\mathscr{E}_{\chi_0}(\mathbb{A}, G)$ is a graded locally finite, $\mathbb{A}$-universal Lie algebra of $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$. Consequently, the graded $\mathbb{A}$-Lie algebra $\mathscr{E}_{\chi_0}(\mathbb{A}, G)$ is torsion-free.*

Proof. — This is similar to the proof of Corollary 1.4. □

Since G is finitely generated, we define:

$$gocha_{\chi_0}(\mathbb{A}, t) := \sum_n c_{\chi_0,n}(\mathbb{A})t^n, \quad \text{where} \quad c_{\chi_0,n}(\mathbb{A}) := \mathrm{rank}_{\mathbb{A}}\mathscr{E}_{\chi_0,n}(\mathbb{A}, G),$$
$$\text{and} \quad a_{\chi_0,n}(\mathbb{A}) := \mathrm{rank}_{\mathbb{A}}(G_{\chi_0,n}(\mathbb{A})/G_{\chi_0,n+1}(\mathbb{A})).$$

2.2. Properties of χ_0-filtrations. — This subpart aims to develop various properties of χ_0-filtations.

Lemma 2.2. — *The modules $\mathscr{E}_{\chi_0}(\mathbb{A}, G)$ and $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$ are graded locally finite $\mathbb{A}[\Delta]$-modules. More precisely, we have:*

$$\mathrm{rank}_{\mathbb{A}}\mathscr{E}_{\chi_0,n}(\mathbb{A}, G)[\chi] = c_{\chi_0,n}(\mathbb{A})\delta_n^{\psi_{\chi_0}(\chi)},$$
$$\mathrm{rank}_{\mathbb{A}}\mathscr{L}_{\chi_0,n}(\mathbb{A}, G)[\chi] = a_{\chi_0,n}(\mathbb{A})\delta_n^{\psi_{\chi_0}(\chi)},$$
$$\textit{where} \quad \delta_n^{\psi_{\chi_0}(\chi)} = 1 \quad \textit{if } n \equiv \psi_{\chi_0}(\chi) \pmod q, \quad \textit{otherwise} \quad \delta_n^{\psi_{\chi_0}(\chi)} = 0.$$

Proof. — Let us denote by $\mathscr{I}_{\chi_0,n}(\mathbb{A}, R) := I_{\chi_0,n}(\mathbb{A}, R)/I_{\chi_0,n+1}(\mathbb{A}, R)$. Remind by [**10**, Lemma 2.15], that $\Delta \subset Aut(F)$ and $\Delta(R) = R$. So $\mathscr{E}_{\chi_0}(\mathbb{A})$ is a graded locally finite $\mathbb{A}[\Delta]$-module, and $\mathscr{I}_{\chi_0,n}(\mathbb{A}, R)$ is stable by Δ. By [**19**, Chapitre I, Résultat 2.3.8.2], we have the following exact sequence:

$$0 \to \mathscr{I}_{\chi_0,n}(\mathbb{A}, R) \to \mathscr{E}_{\chi_0,n}(\mathbb{A}) \to \mathscr{E}_{\chi_0,n}(\mathbb{A}, G) \to 0.$$

Then $\mathscr{E}_{\chi_0}(\mathbb{A}, G)$ and $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$ are graded locally finite $\mathbb{A}[\Delta]$-modules. Let us now study more precisely the $\mathbb{A}[\Delta]$-module structure of $\mathscr{E}_{\chi_0}(\mathbb{A}, G)$ and $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$.

For the structure of $\mathscr{E}_{\chi_0}(\mathbb{A}, G)$: take $u \in \mathscr{E}_{\chi_0,n}(\mathbb{A})$ and write $u := X_{j_1}^{\chi_0^{i_1}} \dots X_{j_u}^{\chi_0^{i_u}}$, with $i_1 + \cdots + i_u = n$. Therefore, for every $\delta \in \Delta$, $\delta(u) = \chi_0^n(\delta)u$. Then, we infer for every χ:

$$(**) \qquad \operatorname{rank}_{\mathbb{A}}\mathscr{E}_{\chi_0,n}(\mathbb{A})[\chi] = \operatorname{rank}_{\mathbb{A}}\mathscr{E}_{\chi_0,n}(\mathbb{A})\delta_n^{\psi_{\chi_0}(\chi)}.$$

Since $\operatorname{rank}_{\mathbb{A}}\mathscr{E}_{\chi_0,n}(\mathbb{A})[\chi] \geqslant \operatorname{rank}_{\mathbb{A}}\mathscr{E}_{\chi_0,n}(\mathbb{A}, G)[\chi]$, we conclude by Equation $(**)$ that:

$$\operatorname{rank}_{\mathbb{A}}\mathscr{E}_{\chi_0,n}(\mathbb{A}, G)[\chi] = c_{\chi_0,n}\delta_n^{\psi_{\chi_0}(\chi)}.$$

For the structure of $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$: note by Lemma 2.1 that $\mathscr{E}_{\chi_0}(\mathbb{A}, G)$ is a graded locally finite $\mathbb{A}[\Delta]$-module, universal $\mathbb{A}$-Lie algebra of $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$. Hence for every χ, and every n:

$$\operatorname{rank}_{\mathbb{A}}\mathscr{E}_{\chi_0,n}(\mathbb{A}, G)[\chi] \geqslant \operatorname{rank}_{\mathbb{A}}\mathscr{L}_{\chi_0,n}(\mathbb{A}, G)[\chi].$$

This allows us to conclude for every χ:

$$\operatorname{rank}_{\mathbb{A}}\mathscr{L}_{\chi_0,n}(\mathbb{A}, G)[\chi] = a_{\chi_0,n}\delta_n^{\psi_{\chi_0}(\chi)}.$$

□

Now, let us compare $(c_{\chi_0,n})_{n\in\mathbb{N}}$, $(a_{\chi_0,n})_{n\in\mathbb{N}}$, $(c_n^{\chi})_{n\in\mathbb{N}}$ and $(a_n^{\chi})_{n\in\mathbb{N}}$.

Proposition 2.3. — *The following inequalities hold:*

$$(6) \qquad c_{\chi_0,qn+i} \leqslant \sum_{j=n}^{qn+i} c_j^{\chi_0^i}, \quad a_{\chi_0,qn+i} \leqslant \sum_{j=n}^{qn+i} a_j^{\chi_0^i},$$

$$(7) \qquad c_n^{\chi} \leqslant \sum_{k=\lceil \frac{n-\psi_{\chi_0}(\chi)}{q} \rceil}^{\lfloor n-\psi_{\chi_0}(\chi)/q \rfloor} c_{\chi_0,qk+\psi_{\chi_0}(\chi)}, \quad a_n^{\chi} \leqslant \sum_{k=\lceil \frac{n-\psi_{\chi_0}(\chi)}{q} \rceil}^{\lfloor n-\psi_{\chi_0}(\chi)/q \rfloor} a_{\chi_0,qk+\psi_{\chi_0}(\chi)}.$$

Proof. — Observe first that the $\mathbb{A}$-Lie algebras $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$, $\mathscr{L}(\mathbb{A}, G)$, $\mathscr{E}_{\chi_0}(\mathbb{A}, G)$ and $\mathscr{E}(\mathbb{A}, G)$ are generated by $\{X_j^{\chi}\}$. We only check inequalities involving c_n (proof of inequalities involving a_n are simlar).

Let us prove inequalities (6).

Take u in $\mathscr{E}_{\chi_0,qn+i}(\mathbb{A}, G)$. Since u is a sum of monomials u_l in $\mathscr{E}_{\chi_0,qn+i}(\mathbb{A}, G)$, we can assume that u is a monomial. So, let us write $u = X_{j_1}^{\chi_0^{i_1}} \dots X_{j_{r_u}}^{\chi_0^{i_{r_u}}}$, where $i_1 + \cdots + i_{r_u} = qn + i$. Consequently for every $\delta \in \Delta$,

$$\delta(u) = {\chi_0}^{i_1}(\delta)X_{j_1}^{\chi_0^{i_1}} \dots {\chi_0}^{i_{r_u}}(\delta)X_{j_{r_u}}^{\chi_0^{i_{r_u}}} \quad \text{thus}$$
$$\delta(u) = {\chi_0}^{i_1+\cdots+i_{r_u}}(\delta)X_{j_1}^{\chi_0^{i_1}} \dots X_{j_{r_u}}^{\chi_0^{i_{r_u}}} = \chi_0^i(\delta)u.$$

Therefore $u \in \mathscr{E}_{r_u}(\mathbb{A}, G)[\chi_0^i]$. To conclude, we need to estimate r_u.

- If $i_l = 1$ for all l, then $r_u = qn + i$.
- If $i_l = q$ for all l, then $qr_u = qn + i$. Therefore, $r_u \geqslant n$.

In any case:

$$n \leqslant r_u \leqslant qn + i.$$

Let us now prove inequalities (7).

Take $u \in \mathscr{E}_n(\mathbb{A}, G)[\chi]$. Since u is a sum of monomials, we can again assume that u is a monomial. Then by Lemma (2.2), we write $u = X_{j_1}^{\chi_0^{i_1}} \dots X_{j_n}^{\chi_0^{i_n}}$, with $i_1 + \cdots + i_n = kq + \psi_{\chi_0}(\chi)$ for some k. Let us see which values can take k:

- if each $i_l = 1$, one obtains $kq + \psi_{\chi_0}(\chi) = n$, and so $k \geqslant \lceil \frac{n-\psi_{\chi_0}(\chi)}{q} \rceil$,
- if each $i_l = q$, one obtains $qn = kq + \psi_{\chi_0}(\chi)$, and so $k \leqslant \lfloor \frac{qn-\psi_{\chi_0}(\chi)}{q} \rfloor$.

In any case:

$$\lceil \frac{n-\psi_{\chi_0}(\chi)}{q} \rceil \leqslant k \leqslant \lfloor n - \psi_{\chi_0}(\chi)/q \rfloor.$$

□

Remark 2.4. — Proposition 2.3 was also given and proved by Anick: Proof of [**2**, Theorem 3].

2.3. Some results on the series $\log(gocha_{\chi_0}(\mathbb{A}, t))$**.** — In this subpart, we obtain information on $(a_{\chi_0,n}(\mathbb{A}))_{n\in\mathbb{N}}$. For this purpose, we study the sequence $(b_{\chi_0,n}(\mathbb{A}))_{n\in\mathbb{N}}$ namely defined by:

$$\log(gocha_{\chi_0}(\mathbb{A}, t)) := \sum_{n\in\mathbb{N}} b_{\chi_0,n} t^n.$$

Theorem 2.5. — *The following equality holds in* $\mathbb{N}[[t]]$*:*

$$gocha_{\chi_0}(\mathbb{A}, t) = \prod_n P(\mathbb{A}, t^n)^{a_{\chi_0,n}},$$

$$\textit{where} \quad P(\mathbb{F}_p, t) := \frac{1-t^p}{1-t}, \quad \textit{and} \quad P(\mathbb{Z}_p, t) := \frac{1}{1-t}.$$

Proof. — By Lemma 2.1, $\mathscr{E}_{\chi_0}(\mathbb{A}, G)$ is a graded locally finite, $\mathbb{A}$-universal Lie algebra of $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$. [**22**, Corollary 2.2] allows us to conclude the case $\mathbb{A} := \mathbb{F}_p$, and [**17**, Proposition 2.5] allows us to conclude the case $\mathbb{A} := \mathbb{Z}_p$. □

Corollary 2.6. — *Let us write* $n = mp^k$*, with* $(m, p) = 1$*, then:*

$$a_{\chi_0,n}(\mathbb{F}_p) = \sum_{r=1}^{k} w_{\chi_0,mp^r}(\mathbb{F}_p), \quad \textit{and} \quad a_{\chi_0,n}(\mathbb{Z}_p) = w_{\chi_0,n}(\mathbb{Z}_p);$$

$$\textit{where} \quad w_{\chi_0,n} := \frac{1}{n}\sum_{m|n} \mu(n/m) b_{\chi_0,m}.$$

Proof. — This proof is similar to the proof of [**21**, Theorem 2.9]. □

Corollary 2.7. — *The following assertions hold:*

(i) If χ *is a non-trivial irreducible character, and there exists an infinite family of primes* $q_i \equiv \psi_{\chi_0}(\chi) \pmod q$ *such that*

$$b_{\chi_0,q_i} > b_{\chi_0,1},$$

then $\mathscr{L}(\mathbb{A}, G)[\chi]$ *is infinite dimensional.*

(ii) If there exists an infinite family of primes $(l_m)_m$ *such that:*

$$b_{\chi_0,ql_m} \geqslant q b_{\chi_0,q} + l_m b_{\chi_0,l_m},$$

then $\mathscr{L}(\mathbb{A}, G)[\mathbb{1}]$ *is infinite dimensional.*

Proof. — This is a consequence of Corollary 2.6. □

Theorem 2.8. — *Assume there exist* $\alpha > 1$ *and a constant* $C \neq 0$ *such that* $b_{\chi_0,n} \underset{n\to\infty}{\sim} C\alpha^n/n$*. Then every eigenspace of* $\mathscr{L}(\mathbb{A}, G)$ *is infinite dimensional.*

Proof. — By Corollary 2.7, we have:

$$a_{\chi_0,q_i} = b_{\chi_0,q_i} - b_{\chi_0,1}/q_i, \qquad \text{and}$$
$$a_{\chi_0,ql_m} = b_{\chi_0,ql_m} - qb_{\chi_0,q} - l_m b_{\chi_0,l_m}.$$

Since, $b_{\chi_0,n} \underset{n\to\infty}{\sim} C\alpha^n/n$, we can find families of primes $\{q_i\}_i$ and $\{l_m\}_m$ where q_i and l_m are sufficiently big, such that: $a_{\chi_0,q_i} > 0$, and $a_{\chi_0,ql_m} > 0$. Therefore by inequalities (6), we extract an infinite subsequence of $(a_n^\chi)_n$ which is strictly positive. □

3. Examples

Recall that $1 \to R \to F \to G \to 1$ denotes a minimal presentation of G, and by [**10**, Lemma 2.15], the group Δ lifts to a subgroup of $Aut(F)$. Keep in mind that $\mathscr{L}(\mathbb{A}, G)$ and $\mathscr{L}_{\chi_0}(\mathbb{A}, G)$ are assumed to be torsion-free over $\mathbb{A}$. Additionally here, G is assumed finitely presented, with cohomological dimension less than or equal to 2.

Consider the following $\mathbb{A}[\Delta]$-modules:

$$R(\mathbb{F}_p) := R/R^p[R;F], \quad \text{and} \quad R(\mathbb{Z}_p) := R/[R;F].$$

Choose χ_0 a non-trivial element of $\mathrm{Irr}(\Delta)$. For every $\chi \in \mathrm{Irr}(\Delta)$, we fix $\{l_j^\chi\}_{1\leqslant j\leqslant r^\chi}$, where $r^\chi := \mathrm{rank}_{\mathbb{A}} R(\mathbb{A})$, a lifting in F of a basis of $R(\mathbb{A})[\chi]$. By [**24**, Corollaire 3, Proposition 42, Chapitre 14], these liftings do not depend on $\mathbb{A}$.

Recall that we defined, using the Magnus isomorphism $\phi_{\mathbb{A}}$ given by (4), the filtered algebras $E(\mathbb{A}, G)$ (in Notations) and $E_{\chi_0}(\mathbb{A}, G)$ (in Subpart 2.1).
Name n_j^χ (resp. $n_{\chi_0,j}^\chi$) the least integer n such that $\phi_{\mathbb{A}}(l_j^\chi - 1)$ is in $E_n(\mathbb{A})\backslash E_{n+1}(\mathbb{A})$ (resp. $E_{\chi_0,n}(\mathbb{A})\backslash E_{\chi_0,n+1}(\mathbb{A})$): this is the degree of l_j^χ in $E(\mathbb{A})$ (resp. $E_{\chi_0}(\mathbb{A})$). We show in Lemma 3.4 that these degrees do not depend on $\mathbb{A}$. Set the series:

$$\chi_{eul}(\mathbb{A}, t) := 1 - dt + \sum_{\chi;1\leqslant j\leqslant r^\chi} t^{n_j^\chi},$$
$$\chi_{eul}^*(\mathbb{A}, t) := 1 - \sum_\chi d^\chi \chi.t + \sum_{\chi;1\leqslant j\leqslant r^\chi} \chi.t^{n_j^\chi},$$
$$\chi_{eul,\chi_0}(\mathbb{A}, t) := 1 - \sum_\chi d^\chi t^{\psi_{\chi_0}(\chi)} + \sum_{\chi;1\leqslant j\leqslant r^\chi} t^{n_{\chi_0,j}^\chi}.$$

3.1. Generalities.

— We give some generalities on groups of cohomological dimension less than or equal to 2.

3.1.1. Computation of some gocha series. — Let us first recall the Lyndon's resolution, which allows us to compute *gocha* series as inverses of polynomials of the form χ_{eul}. A general reference is the article of Brumer [**3**].

Theorem 3.1. — *There exists a filtered $E(\mathbb{A}, G)$-module $M(\mathbb{A})$, such that we have the following exact sequence of filtered $E(\mathbb{A}, G)$-modules:*

$$0 \to M(\mathbb{A}) \to \bigoplus_{\chi;1\leqslant j\leqslant r^\chi} (\phi_{\mathbb{A}}(l_j^\chi - 1))E(\mathbb{A}, G) \to$$
$$\bigoplus_{\chi;1\leqslant j\leqslant d^\chi} (\phi_{\mathbb{A}}(x_j^\chi - 1))E(\mathbb{A}, G) \to E(\mathbb{A}, G) \to \mathbb{A} \to 0.$$

And the cohomological dimension of G is less than or equal to two if and only if $M(\mathbb{A}) = 0$.

There exists a filtered $E_{\chi_0}(\mathbb{A}, G)$-module $M_{\chi_0}(\mathbb{A})$, such that we have the following exact sequence of filtered $E_{\chi_0}(\mathbb{A}, G)$-modules:

$$0 \to M_{\chi_0}(\mathbb{A}) \to \bigoplus_{\chi;1\leqslant j\leqslant r^\chi} (\phi_{\mathbb{A}}(l_j^\chi - 1))E_{\chi_0}(\mathbb{A}, G) \to \bigoplus_{\chi;1\leqslant j\leqslant d^\chi} (\phi_{\mathbb{A}}(x_j^\chi - 1))E_{\chi_0}(\mathbb{A}, G) \to E_{\chi_0}(\mathbb{A}, G) \to \mathbb{A} \to 0.$$

And the cohomological dimension of G is less than or equal to two if and only if $M_{\chi_0}(\mathbb{A}) = 0$.

Remark 3.2. — Theorem 3.1 is true for every filtration over $Al(\mathbb{A}, G)$.

Proof. — Let us define the following $\mathbb{A}$-modules:

$$K(\mathbb{F}_p) := R/R^p[R;R], \quad \text{and} \quad K(\mathbb{Z}_p) := R/[R;R].$$

Notice that $Al(\mathbb{A}, G)$ acts on $K(\mathbb{A})$ via conjugation (see [**13**, Part 7.3]).

By [**3**, Sequence (5.2.2)], we have the following sequence of $Al(\mathbb{A}, G)$-pseudocompact-modules:

$$0 \to K(\mathbb{A}) \to \bigoplus_{\chi,j} \phi_{\mathbb{A}}(x_j^\chi - 1)Al(\mathbb{A}, G) \to Al(\mathbb{A}, G) \to \mathbb{A} \to 0.$$

By [**13**, Theorem 7.7], there exists a $Al(\mathbb{A}, G)$-pseudocompact-module $K'(\mathbb{A})$ such that we have the exact sequence:

$$0 \to K'(\mathbb{A}) \to \bigoplus_{\chi,j} \phi_{\mathbb{A}}(l_j^\chi - 1)Al(\mathbb{A}, G) \to K(\mathbb{A}) \to 0.$$

Furthermore $\mathrm{cd}(G) \leqslant 2$ if and only if $K'(\mathbb{A}) = 0$. Therefore, we obtain the following resolutions:

$$0 \to M(\mathbb{A}) \to \bigoplus_{\chi;1\leqslant j\leqslant r^\chi} (\phi_{\mathbb{A}}(l_j^\chi - 1))E(\mathbb{A}, G) \to \bigoplus_{\chi;1\leqslant j\leqslant d^\chi} (\phi_{\mathbb{A}}(x_j^\chi - 1))E(\mathbb{A}, G) \to E(\mathbb{A}, G) \to \mathbb{A} \to 0,$$

where $M(\mathbb{A})$ is the set $K'(\mathbb{A})$ endowed with its structure of filtered $E(\mathbb{A}, G)$-module, and

$$0 \to M_{\chi_0}(\mathbb{A}) \to \bigoplus_{\chi;1\leqslant j\leqslant r^\chi} (\phi_{\mathbb{A}}(l_j^\chi - 1))E_{\chi_0}(\mathbb{A}, G) \to \bigoplus_{\chi;1\leqslant j\leqslant d^\chi} (\phi_{\mathbb{A}}(x_j^\chi - 1))E_{\chi_0}(\mathbb{A}, G) \to E_{\chi_0}(\mathbb{A}, G) \to \mathbb{A} \to 0,$$

where $M_{\chi_0}(\mathbb{A})$ is the set $K'(\mathbb{A})$ endowed with its structure of filtered $E_{\chi_0}(\mathbb{A}, G)$-module.

Finally, $\mathrm{cd}(G) \leqslant 2 \iff K'(\mathbb{A}) = 0 \iff M(\mathbb{A}) = 0 \iff M_{\chi_0}(\mathbb{A}) = 0.$ □

Let us now compute *gocha* series:

Proposition 3.3. — *We have the following equivalences:*

$$\mathrm{cd}(G) \leqslant 2 \iff gocha(\mathbb{A}, t) = \frac{1}{\chi_{eul}(\mathbb{A}, t)} \iff gocha^*(\mathbb{A}, t) = \frac{1}{\chi_{eul}^*(\mathbb{A}, t)} \iff gocha_{\chi_0}(\mathbb{A}, t) = \frac{1}{\chi_{eul,\chi_0}(\mathbb{A}, t)}.$$

Proof. — One denotes by ρ_j^χ (resp. $\rho_{\chi_0,j}^\chi$) the image of $\phi_{\mathbb{A}}(l_j^\chi-1)$ in $\mathscr{E}_{n_j^\chi}(\mathbb{A})$ (resp. $\mathscr{E}_{n_{\chi_0,j}^\chi}(\mathbb{A})$).

By [**19**, Chapitre I, Formule 2.3.8.2], the functor Grad is exact, then we apply [**19**, Chapitre II, Proposition 3.1.3] and Theorem 3.1, to obtain the following exact sequences of graded locally finite modules:

$$(\star)\quad 0 \to \mathrm{Grad}(M(\mathbb{A})) \to \bigoplus_{\chi;j} \rho_j^\chi \mathscr{E}(\mathbb{A},G) \to \bigoplus_{\chi;j} X_j^\chi \mathscr{E}(\mathbb{A},G) \to \mathscr{E}(\mathbb{A},G) \to \mathbb{A} \to 0,$$

$$(\star\star)\quad 0 \to \mathrm{Grad}(M_{\chi_0}(\mathbb{A})) \to \bigoplus_{\chi;j} \rho_{\chi_0,j}^\chi \mathscr{E}_{\chi_0}(\mathbb{A},G) \to \bigoplus_{\chi;j} X_j^\chi \mathscr{E}_{\chi_0}(\mathbb{A},G) \to \mathscr{E}_{\chi_0}(\mathbb{A},G) \to \mathbb{A} \to 0.$$

From Theorem 3.1 and sequence $(\star)$, we infer:

$$\mathrm{cd}(G) \leqslant 2 \iff M(\mathbb{A}) = 0 \iff \mathrm{Grad}(M(\mathbb{A})) = 0 \iff gocha(\mathbb{A},t) = \frac{1}{\chi_{eul}(\mathbb{A},t)}.$$

Moreover Theorem 3.1 and sequence $(\star\star)$ give us:

$$(8)\quad \mathrm{cd}(G) \leqslant 2 \iff M_{\chi_0}(\mathbb{A}) = 0 \iff \mathrm{Grad}(M_{\chi_0}(\mathbb{A})) = 0 \iff gocha_{\chi_0}(\mathbb{A},t) = \frac{1}{\chi_{eul,\chi_0}(\mathbb{A},t)}.$$

From the choice of the families $\{x_j^\chi\}$ and $\{\rho_j^\chi\}$, we infer that the sequence $(\star)$ is exact in the category of graded locally finite $\mathbb{A}[\Delta]$-modules. This allows us to conclude:

$$\mathrm{cd}(G) \leqslant 2 \iff gocha^*(\mathbb{A},t) = \frac{1}{\chi^*_{eul}(\mathbb{A},t)}.$$

□

3.1.2. Answer to [**21**, Question 2.13]. — Extending and reformulating [**21**, Question 2.13] in our equivariant context, when G is finitely presented and $\mathrm{cd}(G) \leqslant 2$, we show in this Subsubpart that:

> *The series $gocha(\mathbb{A},t)$, $gocha^*(\mathbb{A},t)$ and $gocha_{\chi_0}(\mathbb{A},t)$ do not depend on the ring $\mathbb{A}$?*

Lemma 3.4. — *Assume that $\mathscr{L}(\mathbb{Z}_p,G)$ is torsion-free. Then, for every j and every χ, the integers $n_j^\chi(\mathbb{A})$ do not depend on $\mathbb{A}$. Similarly, if $\mathscr{L}_{\chi_0}(\mathbb{Z}_p,G)$ is torsion-free, then the integers $n_{\chi_0,j}^\chi(\mathbb{A})$ do not depend on $\mathbb{A}$*

Proof. — Let us prove that n_j^χ does not depend on $\mathbb{A}$. Recall that $n_j^\chi(\mathbb{F}_p)$ (resp. $n_j^\chi(\mathbb{Z}_p)$) is the degree of l_j^χ in $E(\mathbb{F}_p)$ (resp. $E(\mathbb{Z}_p)$), and $\rho_j^\chi(\mathbb{F}_p)$ (resp. $\rho_j^\chi(\mathbb{Z}_p)$) denotes the image of $\phi_{\mathbb{F}_p}(l_j^\chi - 1)$ in $\mathscr{E}_{n_j^\chi}(\mathbb{F}_p)$ (resp. $\phi_{\mathbb{Z}_p}(l_j^\chi - 1)$ in $\mathscr{E}_{n_j^\chi}(\mathbb{Z}_p)$). Notice that we have a filtered surjection:

$$E(\mathbb{Z}_p) \overset{(\mathrm{mod}\ p)}{\to} E(\mathbb{F}_p), \quad \text{with kernel} \quad pE(\mathbb{Z}_p).$$

Since the choice of the family $\{l_j^\chi\}_{j,\chi}$ does not depend on $\mathbb{A}$, we infer that $\phi_{\mathbb{Z}_p}(l_j^\chi - 1) \equiv \phi_{\mathbb{F}_p}(l_j^\chi - 1) \pmod p$. Therefore, $n_j^\chi(\mathbb{Z}_p) \leqslant n_j^\chi(\mathbb{F}_p)$.

To show that $n_j^\chi(\mathbb{Z}_p) = n_j^\chi(\mathbb{F}_p)$, it is sufficient to show that for every integer j, and character χ, we have $\rho_j^\chi(\mathbb{Z}_p)$ not in $p\mathscr{E}(\mathbb{Z}_p)$.

From [**7**, Proposition 4.3], we infer the following isomorphism of $E(\mathbb{Z}_p, G)$-modules:

$$K(\mathbb{Z}_p) := R/[R;R] \simeq I(\mathbb{Z}_p, R)/E_1(\mathbb{Z}_p)I(\mathbb{Z}_p, R).$$

Since, G is of cohomological dimension 2, by [**13**, Theorem 7.7], we have

$$K(\mathbb{Z}_p) \simeq \prod_{j,\chi} \phi_{\mathbb{Z}_p}(l_j^\chi - 1)E(\mathbb{Z}_p, G).$$

Introduce

$$\mathscr{I}_n(\mathbb{Z}_p, R) := I_n(\mathbb{Z}_p, R)/I_{n+1}(\mathbb{Z}_p, R), \quad \text{and} \quad \mathscr{I}(\mathbb{Z}_p, R) := \bigoplus_{n\in\mathbb{N}} \mathscr{I}_n(\mathbb{Z}_p, R).$$

Then, we observe that

$$\begin{aligned}\mathrm{Grad}(E_1(\mathbb{Z}_p)I(\mathbb{Z}_p, R)) = \mathrm{Grad}(\prod_{i,\chi} X_i^\chi E(\mathbb{Z}_p)I(\mathbb{Z}_p, R)) &= \bigoplus_{i,\chi} \mathrm{Grad}(X_i^\chi I(\mathbb{Z}_p, R)) \\ &= \bigoplus_{i,\chi} X_i^\chi \mathscr{I}(\mathbb{Z}_p, R) = \mathscr{E}_1(\mathbb{Z}_p)\mathscr{I}(\mathbb{Z}_p, R).\end{aligned}$$

Consequently

$$\mathrm{Grad}(K(\mathbb{Z}_p)) \simeq \bigoplus_{j,\chi} \rho_j^\chi(\mathbb{Z}_p)\mathscr{E}(\mathbb{Z}_p, G) \simeq \mathscr{I}(\mathbb{Z}_p, R)/\mathscr{E}_1(\mathbb{Z}_p)\mathscr{I}(\mathbb{Z}_p, R).$$

Assume now, by contradiction, that there exists one integer j_0 and one character χ_0 such that $\rho_{j_0}^{\chi_0}(\mathbb{Z}_p)$ is in $p\mathscr{E}(\mathbb{Z}_p)$, then there exists $u \in \mathscr{E}(\mathbb{Z}_p)$ such that $\rho_{j_0}^{\chi_0} := pu$. Moreover, since $\mathscr{E}(\mathbb{Z}_p, G)$ is torsion-free, we deduce that u is in $\mathscr{I}(\mathbb{Z}_p, R)$. Therefore, there exist elements g_j^χ in $\mathscr{E}(\mathbb{Z}_p, G)$ such that $u \equiv \sum_{j,\chi} g_j^\chi \rho_j^\chi \pmod{\mathscr{E}_1(\mathbb{Z}_p)\mathscr{I}(\mathbb{Z}_p, R)}$. Consequently:

$$\rho_{j_0}^{\chi_0} := pu \equiv \sum_{j,\chi} p g_j^\chi \rho_j^\chi \pmod{\mathscr{E}_1(\mathbb{Z}_p)\mathscr{I}(\mathbb{Z}_p, R)}.$$

Since the family ρ_j^χ is a basis of the free $\mathscr{E}(\mathbb{Z}_p, G)$-module $\mathscr{I}(\mathbb{Z}_p, R)/\mathscr{E}_1(\mathbb{Z}_p)\mathscr{I}(\mathbb{Z}_p, R)$, we infer $pg_{j_0}^{\chi_0} = 1$. This is impossible since p is not invertible in $\mathscr{E}(\mathbb{Z}_p, G)$.

□

Theorem 3.5. — *Assume that $\mathscr{L}(\mathbb{Z}_p, G)$ is torsion-free, then :*

$$gocha(\mathbb{Z}_p, t) = gocha(\mathbb{F}_p, t), \quad \text{and} \quad gocha^*(\mathbb{Z}_p, t) = gocha^*(\mathbb{F}_p, t).$$

Furthermore, if $\mathscr{L}_{\chi_0}(\mathbb{Z}_p, G)$ is torsion-free, then

$$gocha_{\chi_0}(\mathbb{Z}_p, t) = gocha_{\chi_0}(\mathbb{F}_p, t).$$

Proof. — We apply Proposition 3.3 and Lemma 3.4. □

Remark 3.6. — If we remove the hypothesis that $Aut(G)$ contains a subgroup Δ of order q, then we still have:

$$gocha(\mathbb{Z}_p, t) = gocha(\mathbb{F}_p, t).$$

A criterion to obtain finitely presented groups of cohomological dimension less than or equal to 2 is given by [**17**] when p is odd, and by [**18**] when $p = 2$.

3.1.3. Gocha's series and eigenvalues. — Thanks to Proposition 3.3, we can compute *gocha* series. Then applying Formulae (2) and (3), we obtain an explicit equation relating coefficients a_n and a_n^χ. However, the computation of b_n has complexity n (more precisely it depends on $\{c_m\}_{m\leqslant n}$).

If we consider roots of χ_{eul}, we infer a formula for b_n which depends on the arithmetic complexity of n. The following results are mostly adapted in our context from ideas of Labute ([**15**, Formula (1)]) and Weigel ([**27**, Theorem D]).

Let $\deg(G)$ be the degree of χ_{eul}, and λ_i the eigenvalues of G, written as:

$$\chi_{eul}(t) := \prod_{i=1}^{\deg(G)} (1-\lambda_i t).$$

One denotes by M_n the necklace polynomial of degree n:

$$M_n(t) := \sum_{m|n} \mu(n/m)\frac{t^m}{n}.$$

Let us state [**27**, Theorem D]:

Theorem 3.7. — *Assume $(n,q)=1$ and write $n=mp^k$, with $(m,p)=1$. Then we infer:*

$$a_n(\mathbb{Z}_p) = \sum_{i=1}^{n} M_n(\lambda_i), \quad a_n(\mathbb{F}_p) = \sum_{i=1}^{n}\sum_{j=0}^{k} M_{mp^j}(\lambda_i).$$

Proof. — Weigel showed in the proof of [**27**, Theorem 3.4], that:

$$\sum_{i=1}^{n} M_n(\lambda_i) = w_n.$$

Then we conclude using Theorem 3.5 and Formula (2). □

Let us adapt this result in an equivariant context. By a choice of a primitive q-th root of unity, we have $\mathbb{F}_q \subset \mathbb{F}_p^\times \subset \overline{\mathbb{F}_p}$, the algebraic closure of $\mathbb{F}_p$. Consider δ a non-trivial element in Δ, and evaluate χ_{eul}^* in δ by:

$$\chi_{eul}^*(\delta)(t) := 1 - \sum_{\chi} c_1^\chi \chi(\delta) t + \sum_{\chi;1\leqslant j\leqslant r^\chi} \chi(\delta)t^{n_j^\chi} \in \mathbb{F}_p[t] \subset \overline{\mathbb{F}_p}[t].$$

Define $\{\lambda_{\delta,j}\}_{1\leqslant j\leqslant \deg(G)} \subset \overline{\mathbb{F}_p}$ the eigenvalues of $\chi_{eul}^*(\delta)(t)$. We introduce $\mathscr{F}(\Delta,\overline{\mathbb{F}_p})$ the $\overline{\mathbb{F}_p}$-algebra of functions from Δ to $\overline{\mathbb{F}_p}$ and:

$$\eta_j\colon \Delta \to \overline{\mathbb{F}_p}; \quad \delta \mapsto \lambda_{\delta,j}.$$

Therefore, we infer:

$$\chi_{eul}^*(t) := \prod_{j=1}^{\deg(G)} (1-\eta_j t) \in \mathscr{F}(\Delta,\overline{\mathbb{F}_p})[t].$$

Consequenlty, if we apply the log function to the previous equality, we obtain:

$$b_m^* := \sum_{\chi\in \mathrm{Irr}(\Delta)} b_m^\chi \chi = \frac{\eta_1^m + \cdots + \eta_{\deg(G)}^m}{m}.$$

Let us define for every $\eta \in \mathscr{F}(\Delta, \overline{\mathbb{F}_p})$:

$$M_n^*(\eta) := \sum_{m|n} \frac{1}{n}\mu(n/m)\eta^{m,(n/m)}, \quad \text{where} \quad \eta^{m,(u)}(\delta) = \eta(\delta^u)^m.$$

Proposition 3.8. — *Let us assume q divides $p-1$ and $(n,q)=1$. Write $n = mp^k$, with $(m,p)=1$, then:*

$$a_n(\mathbb{Z}_p)^* := \sum_\chi a_n^\chi(\mathbb{Z}_p)\chi = \sum_{j=1}^{\deg(G)} M_n^*(\eta_j), \quad \textit{and}$$

$$a_n(\mathbb{F}_p)^* := \sum_\chi a_n^\chi(\mathbb{F}_p)\chi = \sum_{j=1}^{\deg(G)} \sum_{i=0}^{k} M_{mp^i}^*(\eta_j),$$

the equality is in the $\overline{\mathbb{F}_p}$-algebra $\mathscr{F}(\Delta, \overline{\mathbb{F}_p})$.

Proof. — Let us remind that $b_n^* := \sum_\chi b_n^\chi \chi$. After making the following change of variable: $\gamma = \chi^{n/m}$, we observe that for every δ in Δ, we have

$$b_m^{*1,(n/m)}(\delta) := b_m^*(\delta^{n/m}) = \sum_\chi b_m^\chi \chi(\delta^{n/m}) = \sum_{\chi\in \mathrm{Irr}(\Delta)} b_m^\chi \chi(\delta)^{n/m} = \sum_{\gamma \in \mathrm{Irr}(\Delta)} b_m^{\gamma^{m/n}} \gamma(\delta).$$

Consequently, $b_m^{*1,(n/m)} = \sum_\chi b_m^{\chi^{m/n}}\chi$. Since $mb_m^* = \eta_1^m + \cdots + \eta_{\deg(G)}^m$, we obtain:

$$mb_m^{*1,(m/n)} = \left(\eta_1^m + \cdots + \eta_{\deg(G)}^m\right)^{(n/m)} = \eta_1^{m,(n/m)} + \cdots + \eta_{\deg(G)}^{m,(n/m)}.$$

Using Formula (3), the conclusion follows. □

Remark 3.9. — Filip ([**6**, Formula (4.8)]) and Stix ([**25**, Formula (14.16)]) also obtained Proposition 3.8 for some groups defined by one quadratic relation. They computed explicitely the functions η_j.

Example 3.10. — Let us illustrate Proposition 3.8, with Example 1.

When splitting χ_{eul}^* into eigenvalues, we obtain:

$$\chi_{eul}^*(t) = (1-\eta_1 t)(1-\eta_2 t) = 1 - (\chi_0 + \chi_0^2 + \chi_0^3)t + \chi_0^3 t^3,$$

Moreover, $\eta_1\eta_2 = \chi_0^3$ and $\eta_1 + \eta_2 = \chi_0 + \chi_0^2 + \chi_0^3$ (as functions). Therefore, if we apply Proposition 3.8, we get:

$$\begin{aligned} a_2^* := \sum_\chi a_2^\chi &= \frac{\eta_1^2 + \eta_2^2 - \eta_1^{(2)} - \eta_2^{(2)}}{2} = \frac{(\eta_1+\eta_2)^2 - 2\eta_1\eta_2 - (\eta_1+\eta_2)^{(2)}}{2} \\ &= \frac{\chi_0^2 + \chi_0^4 + \chi_0^6 + 2\chi_0^3 + 2\chi_0^4 + 2\chi_0^5 - 2\chi_0^3 - \chi_0^2 - \chi_0^4 - \chi_0^6}{2} = \chi_0^4 + \chi_0^5. \end{aligned}$$

Let us now compute a_3^χ. For this purpose, we first observe that

$$\begin{aligned} \eta_1^3 + \eta_2^3 &= (\chi_0 + \chi_0^2 + \chi_0^3)^3 - 3(\chi_0 + \chi_0^2 + \chi_0^3)\chi_0^3 \\ &= \chi_0^9 + 3\chi_0^8 + 6\chi_0^7 + 4\chi_0^6 + 3\chi_0^5 + \chi_0^3. \end{aligned}$$

Therefore, we have:

$$a_3^* := \sum_\chi a_3^\chi = \frac{\eta_1^3 + \eta_2^3 - \eta_1^{(3)} - \eta_2^{(3)}}{3} = \frac{\eta_1^3 + \eta_2^3 - (\eta_1+\eta_2)^{(3)}}{3} = \chi_0^5 + \chi_0^6 + 2\chi_0^7 + \chi_0^8.$$

Let us conclude this subpart by proving Theorem B given in our introduction.

Theorem 3.11. — *Assume that $\mathscr{L}(\mathbb{A}, G)$ is infinite dimensional and for some χ_0 that $L_{\chi_0}(G)$ is reached for a unique eigenvalue λ_{χ_0} such that:*

(i) λ_{χ_0} *is real,*

(ii) $L_{\chi_0}(G) = \lambda_{\chi_0} > 1$.

Then every eigenspace of $\mathscr{L}(\mathbb{A}, G)$ is infinite dimensional.

Proof. — We study the asymptotic behaviour of $(b_{\chi_0,n})_{n\in\mathbb{N}}$. By Proposition 3.3, we have:

$$gocha_{\chi_0}(t) := \frac{1}{\chi_{eul,\chi_0}(t)}.$$

Let us denote by $\{\lambda_1; \dots; \lambda_u\}$ the real χ_0-eigenvalues of G and $\{\beta_1 e^{i\pm\theta_1}; \dots; \beta_v e^{i\pm\theta_v}\}$ the polar forms of non real χ_0-eigenvalues of G. Without loss of generality, assume that $\lambda_{\chi_0} := \lambda_1$. Let us write

$$\chi_{eul,\chi_0}(t) := \prod_{i=1}^{u}(1-\lambda_i t)\prod_{j=1}^{v}(1-\beta_j e^{i\theta_j}t)(1-\beta_j e^{-i\theta_j}t).$$

Then, we obtain:

$$\log(\chi_{eul,\chi_0}(t)) = \sum_{n\in\mathbb{N}} \frac{\sum_{i=1}^{u}\lambda_i^n + \sum_{j=1}^{v}\beta_j^n(e^{in\theta_j}+e^{-in\theta_j})}{n}t^n.$$

Thus $b_{\chi_0,n} \underset{n\to\infty}{\sim} C\lambda_1^n/n$, for some $C > 0$. We conclude by Theorem 2.8. □

3.2. Group Theoretical examples. —

3.2.1. Free pro-p groups. — In this subpart, assume that G is a free finitely generated pro-p group. Observe that $\mathscr{L}(\mathbb{Z}_p, G)$ and $\mathscr{L}_{\chi_0}(\mathbb{Z}_p, G)$ are torsion-free.

Theorem 3.12. — *Assume that G is a noncommutative free pro-p group, then every eigenspace of $\mathscr{L}(\mathbb{A}, G)$ is infinite dimensional.*

Proof. — Let us fix a non-trivial character $\chi_0 \in \mathrm{Irr}(\Delta)$, such that $d^{\chi_0} \leqslant d^{\chi}$ for every non-trivial χ. Then we have $\chi_{eul,\chi_0}(t) := 1 - \sum_{i=1}^{q} d^{\chi_0^i} t^i$. Set s a minimal positive real root of χ_{eul,χ_0}. We will show that s is the unique root of minimal absolute value of χ_{eul,χ_0}.

We have:

$$0 = 1 - \sum_{i=1}^{q} d^{\chi_0^i} s^i \leqslant 1 - d^{\chi_0} s \sum_{i=0}^{q-2} s^i - d^{\mathbb{1}} s^q \leqslant 1 - d^{\chi_0} s - d^{\mathbb{1}} s^q.$$

Then $d^{\chi_0}s + d^{\mathbb{1}}s^q \leqslant 1$. Thus $s \leqslant \min\{1/d^{\chi_0}; (1/d^{\mathbb{1}})^{1/q}\}$, so $0 < s < 1$.

If we denote by z a complex root (not in $]0;1[$) of χ_{eul,χ_0}, then we notice by the triangle inequality, that $\chi_{eul,\chi_0}(|z|) < \chi_{eul,\chi_0}(z) = 0$. Therefore $|z| > s$.

Consequently, χ_{eul,χ_0} admits a unique root s of minimal absolute value which is in $]0;1[$. Therefore, by Theorem B, we conclude. □

Let us give some examples.

Example 3.13. — Consider $\Delta := \mathbb{Z}/2\mathbb{Z}$, and fix χ_0 the non-trivial irreducible character of Δ over $\mathbb{A}$. Assume that G is a free pro-p group with two generators $\{x, y\}$, and Δ acts on G by: $\delta(x) = x$, $\delta(y) = y^{-1}$. Then following our notations, we have: $x = x^{\mathbb{1}}$, and $y = x^{\chi_0}$. Observe that $Al(\mathbb{A}, G)$ is a free algebra on two variables over $\mathbb{A}$.
Let us first compute some coefficients a_n^{χ}, with Formula (3). We have:

$$gocha^*(\mathbb{A}, t) := \frac{1}{1-(1+\chi_0).t}, \quad \text{and} \quad \log(gocha^*(\mathbb{A}, t)) := \sum_n \frac{(1+\chi_0)^n}{n} t^n.$$

So

$$c_{2n}^{\mathbb{1}} = c_{2n}^{\chi_0} = 2^{2n-1}, \quad c_{2n+1}^{\chi_0} = c_{2n+1}^{\mathbb{1}} = 2^{2n},$$

$$b_{2n+1}^{\chi_0} = b_{2n+1}^{\chi_0} = \frac{2^{2n}}{2n+1}, \quad \text{and} \quad b_{2n}^{\mathbb{1}} = b_{2n}^{\chi_0} = \frac{2^{2n-1}}{2n}.$$

Assume for instance $p \neq 3$, then one obtains:

$$a_3^{\chi_0} = \frac{2^2-1}{3} = 1, \quad \text{and} \quad a_3^{\mathbb{1}} = 1.$$

Observe by Theorem 3.12, that every eigenspace of $\mathscr{L}(\mathbb{A}, G)$ is infinite.

Example 3.14. — Again, take $\Delta := \mathbb{Z}/2\mathbb{Z}$ and χ_0 the unique non-trivial $\mathbb{A}$-irreducible character of Δ. Assume G is free generated by $\{x_1^{\chi_0}; \dots; x_d^{\chi_0}\}$.

First, we compute some coefficients of $(c_n^{\chi})_n$ and $(a_n^{\chi})_n$. Observe:

$$gocha^*(\mathbb{A}, t) := \frac{1}{1-d\chi_0 t}, \quad \text{and} \quad gocha_{\chi_0}(\mathbb{A}, t) := \frac{1}{1-dt}.$$

Then $c_{2n}^{\mathbb{1}} = d^{2n}$, $\quad c_{2n}^{\chi_0} = 0$, $\quad c_{2n+1}^{\chi_0} = d^{2n+1}$, $\quad$ and $\quad c_{2n+1}^{\mathbb{1}} = 0$.
Moreover,

$$\log(gocha^*(\mathbb{A}, t)) := \sum_n \frac{(d\chi_0)^n}{n} t^n, \quad \log(gocha_{\chi_0}(\mathbb{A}, t)) := \sum_{n \in \mathbb{N}} \frac{d^n}{n} t^n.$$

So, $b_{2n+1}^{\chi_0} := d^{2n+1}/(2n+1)$, $\quad b_{2n}^{\chi_0} = 0$, $\quad b_{2n}^{\mathbb{1}} = d^{2n}/(2n)$, $\quad$ and $\quad b_{2n+1}^{\mathbb{1}} = 0$.
For instance, if we apply Formula (3), one obtains when $p \neq 3$:

$$a_3^{\chi_0} = \frac{d^3-d}{3}, \text{ and } a_3^{\mathbb{1}} = 0.$$

If we apply Proposition 3.8, we obtain:

$$a_2^{\chi_0} = 0, \quad \text{and} \quad a_2^{\mathbb{1}} = \frac{d^2-d}{2}.$$

Observe that $c_{\chi_0,n} = d^n$ and $b_{\chi_0,n} := d^n/n$. Theorem 3.12, allows us to check that every eigenspace of $\mathscr{L}(\mathbb{A}, G)$ is indeed infinite dimensional.

3.2.2. Non-free case. — Let us now construct some non-free examples that illustrate Theorem B. For this purpose, consider Δ a subgroup of $Aut(F)$. We construct here a finitely presented pro-p quotient G of F, such that Δ induces a subgroup of $Aut(G)$.

We remind that F is the free pro-p group generated by $\{x_j^{\chi}\}_{\chi \in \mathrm{Irr}(\Delta); 1 \leqslant j \leqslant d^{\chi}}$ and define $\mathscr{F}$ the free abstract group generated by the family $\{x_j^{\chi}\}_{\chi; j}$. Assume also that the action of Δ is diagonal over $\{x_j^{\chi}\}$, i.e. for all δ in Δ, $\delta(x_j^{\chi}) = (x_j^{\chi})^{\chi(\delta)}$.

***Definition 3.15* (Comm-family).** — The family $(l_j)_{j\in[\![1;r]\!]}\subset\mathscr{F}$ is said to be a comm-family if:

$$l_j:=\prod_{l=1}^{\eta_j}u_{j,\gamma_l}^{\alpha_{j,\gamma_l}}\in F,$$

where γ_l and α_{j,γ_l} are integers, and u_{j,γ_l} is a γ_l-th commutator on $\{x_j^\chi\}_{\chi;j}$, i.e. $u_{j,\gamma_l}:=[x_1;\dots;x_{\gamma_l}]$ where $x_i\in\{x_j^\chi\}_{\chi;j}$.

Proposition 3.16. — *Let $(l_j)_{j\in[\![1;r]\!]}$ be a comm-family, and denote by R its normal (topological) closure in F. Then for all δ in Δ, $\delta(R)=R$ thus Δ induces a subgroup of $Aut(F/R)$.*

Proof. — First of all, if u and v are elements in F, we write $u^v:=v^{-1}uv$.
Assume $[x;y]\in R$, where x and y are elements in $\{x_j^\chi\}_{\chi;j}$. Observe the following identity:

$$1=[x;yy^{-1}]=[x;y^{-1}][x;y]^{y^{-1}}.$$

Therefore $[x;y^{-1}]$ is in R. Remark also for all integers a:

$$[x;y^a]=[x;y^{a-1}][x;y]^{y^{a-1}}.$$

Thus by induction, we see that for all $a\in\mathbb{Z}$, the commutator $[x;y^a]$ is in R.
Finally, for all integers b, we also have:

$$[x^b;y]=[x;y]^{x^{b-1}}[x^{b-1};y].$$

We conclude as before that $[x^b;y]\in R$, for all integers b.
Then $\delta(R)=R$, for every $\delta\in\Delta$. □

Example 3.17. — Here assume q is an odd prime that divides $p-1$. Take F a free pro-p group with three generators: $\{x_1^{\chi_0},x_1^{\chi_0^2},x_1^{\chi_0^3}\}$. Assume also that Δ acts diagonally on the previous set.

Consider R the closed normal subgroup of F generated by commutators $l_1:=[x_1^{\chi_0};x_1^{\chi_0^2}]$ and $l_2:=[x_1^{\chi_0};x_1^{\chi_0^3}]$. By Proposition 3.16, the group Δ induces a subgroup of $Aut(G)$. Since G is mild (see for instance [**7**]), we have $\mathrm{cd}(G)=2$ and:

$$gocha_{\chi_0}(\mathbb{F}_p,t)=\frac{1}{\chi_{eul,\chi_0}(\mathbb{F}_p,t)}=\frac{1}{1-t-t^2+t^4}.$$

Thus by Theorem B, we conclude that every eigenspace of $\mathscr{L}(\mathbb{F}_p,G)$ is infinite dimensional.

3.3. FAB quadratic mild examples. — Let K be a quadratic imaginary extension over $\mathbb{Q}$, with class number coprime to p. Denote by $S:=\{\mathfrak{p}_1;\dots;\mathfrak{p}_d\}$ a finite set of tame places of K, i.e. for $\mathfrak{p}\in S$, $N_{\mathrm{K}/\mathbb{Q}}(\mathfrak{p})\equiv 1 \pmod p$, and assume that S is stable by Δ. We define K_S the p-maximal unramified extension of K outside S. Set $G:=\mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$ and $\Delta:=\mathrm{Gal}(\mathrm{K}/\mathbb{Q})$. Again, fix χ_0 the non-trivial character of Δ over $\mathbb{F}_p$. The group Δ acts on G, and thanks to Class Field Theory, the group G has the FAB property: every open subgroup has finite abelianization.

Write $U_\mathfrak{p}$ for the unit group of the completion of K at the place $\mathfrak{p}\in S$. We define the element $X_\mathfrak{p}\in\mathscr{E}_1(\mathbb{F}_p,G)$ as the image, given by Class Field Theory, of a generator of $U_\mathfrak{p}/U_\mathfrak{p}^p$. Then (see for instance [**23**, Theorem 2.6]), the set $\{X_\mathfrak{p}\}_{\mathfrak{p}\in S}$ is a basis of $\mathscr{E}_1(\mathbb{F}_p,G)$.

Denote by $x_{\mathfrak{p}}$ an element in G that lifts $X_{\mathfrak{p}}$. We introduce F, the free pro-p group generated by $x_{\mathfrak{p}}$. Koch [**13**, Chapter 11] gave a presentation of G, with generators $\{x_{\mathfrak{p}}\}_{p\in S}$ and relations $\{l_{\mathfrak{p}}\}_{\mathfrak{p}\in S}$ verifying:

$$l_{\mathfrak{p}_i} \equiv \prod_{j\neq i}[x_{\mathfrak{p}_i}, x_{\mathfrak{p}_j}]^{a_j(i)} \pmod{F_3(\mathbb{F}_p)}, \quad \text{where} \quad a_j(i) \in \mathbb{Z}/p\mathbb{Z}.$$

The element $a_j(i)$ is zero if and only if the prime $\mathfrak{p}_i$ splits in $\mathrm{k}^p_{\{\mathfrak{p}_j\}}/\mathrm{k}$, where $k^p_{\{\mathfrak{p}\}}$ is the (unique) cyclic extension of degree p of k unramified outside $\mathfrak{p}$. This is equivalent to

$$p_i^{(p_j-1)/p} \equiv 1 \pmod{p_j},$$

where p_i is a prime in $\mathbb{Q}$ below $\mathfrak{p}_i$.

From now, we assume that this presentation is **mild and quadratic** (the relations are all of weight 2), which means that we have the following isomorphisms of $\mathbb{F}_p[\Delta]$-modules:

$$\mathscr{E}_1(\mathbb{F}_p) = \bigoplus_{i=1}^{d} X_{\mathfrak{p}_i}\mathbb{F}_p, \quad \text{and} \quad R(\mathbb{F}_p) \simeq \bigoplus_{i=1}^{d}\left(\sum_{j\neq i} a_j(i)[X_{\mathfrak{p}_j}; X_{\mathfrak{p}_i}]\right)\mathbb{F}_p.$$

Denote by i (resp. s), the number of inert or totally ramified (resp. totally split) primes below S in $\mathbb{Q}$, then $d = r = |S| := i + 2s$. Recall that for every χ:

$$d^\chi := \dim_{\mathbb{F}_p} \mathscr{E}_1(\mathbb{F}_p)[\chi], \quad \text{and} \quad r^\chi := \dim_{\mathbb{F}_p} R(\mathbb{F}_p)[\chi].$$

By [**9**, Theorem 1] and Class Field Theory, we obtain:

$$d^{\mathbb{1}} = i + s \quad (\text{resp. } r^{\mathbb{1}} = i + s) \quad \text{and} \quad d^{\chi_0} = s \quad (\text{resp. } r^{\chi_0} = s).$$

Proposition 3.18. — *We have the following equalities of series:*

$$gocha^*(\mathbb{F}_p, t) := \frac{1}{1 - (i + s + s\chi_0)t + (i + s + s\chi_0)t^2},$$

$$gocha_{\chi_0}(\mathbb{F}_p, t) := \frac{1}{1 - st - it^2 + (s + i)t^4}.$$

Consequently, the action of Δ on G is not trivial if and only if at least one place above S in $\mathbb{Q}$ totally splits in K.

Proof. — Here, the relations have all weight 2, so:

$$\chi^*_{eul}(t) := 1 - (d^{\mathbb{1}} + d^{\chi_0}\chi_0)t + (r^{\mathbb{1}} + r^{\chi_0}\chi_0)t^2 = 1 - (i + s + s\chi_0)t + (i + s + s\chi_0)t^2.$$

Since the presentation is mild, we conclude using Proposition 3.3. □

Remark 3.19. — Before giving examples, let us add some complements.

- The $\mathbb{F}_p[\Delta]$-module structure of $\mathscr{E}_1(\mathbb{F}_p)$ (or $R(\mathbb{F}_p)$) gives us the integers i and s.
- If every place $\mathfrak{p}$ above S is inert or totally ramified in K, then $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ and $G := \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$ admit the same number of generators. Then Gras [**9**, Theorem 1], showed that $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ and G are isomorphic, so the action of Δ over G is trivial.
- Assume now that all places in $\mathbb{Q}$ below a set of primes S are totally split in K. If $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ is mild, Rougnant in [**23**, Théorème 0.3] gave a criterion to also obtain $\mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$ mild.

Example 3.20. — We give explicit arithmetic examples where G is mild and defined by quadratic relations:

1. We study the following example given by [**26**, Example 3.2]: let $p = 3$, $\mathrm{K} := \mathbb{Q}(i)$, and consider the set of primes: $S := \{q_1 := 229, q_2 := 241\}$. These primes totally split in K. and the places above S (in K) are given by:
$$S := \{\mathfrak{p}_1 := (2+15i), \overline{\mathfrak{p}_1} := (2-15i), \mathfrak{p}_2 := (4+15i), \overline{\mathfrak{p}_2} := (4-15i)\}.$$
The group $G := \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$ is mild quadratic. Then by Proposition 3.18:
$$gocha^*(\mathbb{F}_p, t) = \frac{1}{1-(2+2\chi_0)t+(2+2\chi_0)t^2}, \quad \text{and} \quad gocha_{\chi_0}(\mathbb{F}_p, t) = \frac{1}{1-2t+2t^4}.$$
However, the polynomial $1-2t+2t^4$ admits only non real roots, so we can not apply Theorem B.
Observe by [**13**, Example 11.15], that the group $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ is finite.
2. [**23**, Part 6]: Take $p = 3$, $\mathrm{K} := \mathbb{Q}(\sqrt{-5})$, and $S := \{61; 223; 229; 481\}$. The Class group of K is $\mathbb{Z}/2\mathbb{Z}$, the primes in S are totally split in K, and the groups $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ and $G := \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$ are both mild quadratic. Therefore, by Proposition 3.18, we obtain:
$$gocha^*(\mathbb{F}_p, t) = \frac{1}{1-(4+4\chi_0)t+4\chi_0t^2} \quad \text{and} \quad gocha_{\chi_0}(\mathbb{F}_p, t) = \frac{1}{1-4t+4t^4}.$$
By Theorem B, the graded spaces $\mathscr{L}(\mathbb{F}_p, G)[\chi_0]$ and $\mathscr{L}(\mathbb{F}_p, G)[\mathbb{1}]$ are both infinite dimensional.
3. We enrich the example given in [**11**, Part 2.1]: Consider $p = 3$, $\mathrm{K} := \mathbb{Q}(\sqrt{-163})$, and $T := \{31, 19, 13, 337, 7\}$. The class group of K is trivial, $\mathrm{Gal}(\mathbb{Q}_T/\mathbb{Q})$ is mild, and the primes in T are inert in K. Therefore by [**9**, Theorem 1], the group $\mathrm{Gal}(\mathrm{K}_T/\mathrm{K})$ is mild (in fact, it has the same linking coefficients as $\mathrm{Gal}(\mathbb{Q}_T/\mathbb{Q})$).
Observe that 43 is totally split in K, so we take $\{\mathfrak{p}_6, \overline{\mathfrak{p}_6}\}$ to be the primes in K above 43. Consider now $S := T \cup \{\mathfrak{p}_6; \overline{\mathfrak{p}_6}\}$. By [**26**, Corollary 4.3], the group $G := \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$ is mild quadratic. Proposition 3.18 gives us
$$gocha^*(\mathbb{F}_p, t) := \frac{1}{1-(6+\chi_0)t+(6+\chi_0)t^2}, \quad \text{and} \quad gocha_{\chi_0}(\mathbb{F}_p, t) := \frac{1}{1-t-5t^2+6t^4}.$$
Therefore, by Theorem B, the graded spaces $\mathscr{L}(\mathbb{F}_p, G)[\mathbb{1}]$ and $\mathscr{L}(\mathbb{F}_p, G)[\chi_0]$ are infinite dimensional.

Remark on lower p-central series and mild groups

Assume here that G is a finitely presented pro-p group, and q divides $p-1$. We define the lower p-central series of G by:
$$G_{\{1\}} := G, \quad \text{and} \quad G_{\{n+1\}} := G_{\{n\}}^p[G_{\{n\}}; G].$$
Remark that $\bigoplus_{n\in\mathbb{N}}(G_{\{n\}}/G_{\{n+1\}})$ is an $\mathbb{F}_p[t][\Delta]$-module, where $\mathbb{F}_p[t]$ is the ring of polynomials over $\mathbb{F}_p$.

Furthermore, if we assume G mild (see [**17**, Definition 1.1]), Labute showed in [**17**, Part 4], that the lower p-central series come from the filtered algebra defined by $Al(\mathbb{Z}_p, G)$ endowed with the filtration induced by $\{Al_{\{n\}}(G) := \ker(Al(\mathbb{Z}_p, G) \to \mathbb{F}_p)^n\}_{n\in\mathbb{N}}$. Additionally, the set $\bigoplus_{n\in\mathbb{N}}(G_{\{n\}}/G_{\{n+1\}})$ is a free $\mathbb{F}_p[t]$-module. Since G is finitely generated, we introduce:
$$a_{\{n\}}^\chi := \mathrm{rank}_{\mathbb{F}_p}(G_{\{n\}}/G_{\{n+1\}})[\chi], \quad \text{and} \quad c_{\{n\}}^\chi := \mathrm{rank}_{\mathbb{F}_p}(Al_{\{n\}}(G)/Al_{\{n+1\}}(G))[\chi].$$

If we replace $a_n(\mathbb{Z}_p)$ (resp. $c_n(\mathbb{Z}_p)$) by $a_{\{n\}}$ (resp. $c_{\{n\}}$), then the results of this paper can be adapted for lower p-central series. Moreover, extending [**17**, Corollary 2.7] in an equivariant context, we can deduce a relation between the coefficients c_n^χ and $a_{\{n\}}^\chi$.

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